Rho-form for fibrations

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February 1, 2008

Abstract

In this paper we construct the generalization of the Cheeger-Gromov L^2-rho-invariant in the case of families. It is in fact a differential form: given a family of Dirac-type operators along the fibres of a fibration $M \to B$ and a fibration $\tilde{M} \to B$ of normal Γ-coverings of the fibres, the *rho-form* is defined as the difference between the $\hat{\eta}$-form of Bismut and Cheeger and an L^2-eta-form, $\hat{\eta}_{(2)}$-form, whose meaning is here given.

Some hypothesis on the spectrum of the family of operators on the covering fibres must be made, to construct $\hat{\eta}_{(2)}$. The delicate point is in fact in the $t \to \infty$-asymptotic of the heat operator for the Bismut superconnection on the covering. First we consider a strong hypothesis (unifom invertibility for the two families of operators): in this case $\hat{\eta}_{(2)}$ is well defined. This situation occurs for example in the case of a fibration of spin manifolds with vertical metric of positive scalar curvature. Here we also prove that rho-form is constant on the connected components of the space of metrics on the vertical tangent bundle which are positive scalar curvature along the fibres. After that, following the ideas of [HL], we study the case of some weaker regularity hypothesis for the spectrum of the family of operators on coverings. We find out that if the Novikov-Shubin exponent β is greater than $3(\dim B + 1)$, then the rho-form is well defined and closed.

Contents

1 Introduction

Consider a family of compact manifolds $\{M_z, z \in B\}$, fibres of a smooth fiber bundle $\pi : M \to B$. Let $T(M/B) \subset TM$ be the vertical tangent bundle with a given metric $g_{M/B}$ and fix a connection, i.e. a choice of a splitting $TM = T(M/B) \oplus T_H M$.

Let : $E \to M$ be a vector bundle of vertical Clifford modules, i.e modules for the Clifford algebra bundle $\mathrm{Cl}(T^*M/B, g_{M/B})$, with Clifford connection ∇^E on E. Then we are given a family of Dirac operators $\mathcal{D} = (D_b)_{b\in B}$

$$D_b : \mathcal{C}^\infty(M_b, E_b) \to \mathcal{C}^\infty(M_b, E_b)$$

Suppose that $\dim \operatorname{Ker} D_z$ is constant in z: then one of the results of the heat-kernel proof by Bismut of the family index theorem is that the so-called ***eta-form*** of the family is well defined (see theorem A.2, page 26 in appendix): it is given in the case of even dimensional fibre by

$$\hat{\eta}(\mathcal{D}) = \frac{1}{\sqrt{\pi}} \int_0^\infty \operatorname{Str}\left(\frac{d\mathbb{B}_s}{ds} e^{-\mathbb{B}_s^2}\right) ds \quad \in \mathcal{C}^\infty(B, \Lambda^{even} T^*B) \tag{1}$$

where $\mathbb{B}$ is the so called Bismut superconnection, and in the case of odd dimensional fibre by

$$\hat{\eta}(\mathcal{D}) = \frac{1}{\sqrt{\pi}} \int_0^\infty \operatorname{Tr}_{\mathbb{C}l(1)}\left(\frac{d\mathbb{B}_s}{ds} e^{-\mathbb{B}_s^2}\right) ds \quad \in \mathcal{C}^\infty(B, \Lambda^{odd} T^*B) \tag{2}$$

where $\mathbb{B}$ is the so called $\mathbb{C}l(1)$ Bismut superconnection. When B is a single point $\{0\}$, and the family reduces to a single operator $D = D_0$, then $\hat{\eta}(D_0)$ reduces to the Atiyah-Patodi-Singer η-invariant of the single operator

$$\eta(D) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-\frac{1}{2}} \operatorname{TR}(De^{-tD^2}) dt$$

In fact the eta-form $\hat{\eta}$ plays a crucial role in the family index theorem for manifolds with boundary ([BC] [MP]).

If B is a point and $\tilde{M}_0 \to M_0$ is a normal Γ-covering of the manifold Z, then D can be lifted to a Γ-invariant operator $\tilde{D}$. By means of a von Neumann trace, Cheeger and Gromov defined (in [CG]) the L^2-eta invariant

$$\eta_{(2)}(\tilde{D}) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-\frac{1}{2}} \operatorname{TR}_\Gamma(\tilde{D} e^{-t\tilde{D}^2}) dt \tag{3}$$

and they considered the L^2-rho invariant, defined as the difference

$$\rho_{(2)}(D) = \eta(D) - \eta_{(2)}(\tilde{D})$$

There are two particular Dirac-type operators, whose L^2-rho invariants have many interesting applications to geometry: they are the Dirac operator on a spin manifold, and the signature operator. Let's recall some results which motivate us to generalize $\rho_{(2)}$ to an analogous quantity for families.

(a) Let M_0 be a spin manifold, and $D = \not{D}_g$ the Dirac operator, w.r.t. a Riemannian metric g. Let $r : M \to B\Gamma$ be a map classifying a Γ-normal covering. Let $\mathcal{R}^+(M_0)$ be the space of metrics on M_0 which are positive scalar curvature. Observe that $\rho_{(2)}$ depends on the metric on M_0.

- **(a.1)** *$\rho_{(2)}(\not{D})$ is constant on the connected components of $\mathcal{R}^+(M_0)$.* (It follows from Lichnerowicz formula and index theorems on manifolds with boundary.)
- **(a.2)** *If Γ is torsion-free and satisfies the Baum-Connes conjecture for the maximal C^*-algebra, then $\rho_{(2)}(\not{D}_g)$=0 if $g \in \mathcal{R}^+(Z)$.* (Piazza-Schick [PS1])
- **(a.3)** *If M is a spin manifold of dimension $4k+3$, $k > 0$, if $\mathcal{R}^+(M)$ is non void and Γ has torsion, then M has infinitely many different Γ-bordism-classes of metrics with positive scalar curvature: these metrics are distinguished by $\rho_{(2)}(\not{D}_g)$* (Theorem 1.3 in [PS2]).

(b) Consider a Riemannian manifold M and let $D = D^{sign}$ be the signature operator.

- **(b.1)** *$\rho_{(2)}$ does not depend on the metric on M.* (Observed by Cheeger and Gromov in [CG2].)
- **(b.2)** *If Γ is torsion-free and satisfies the Baum-Connes conjecture, then $\rho_{(2)}$ depends only on the oriented Γ-homotopy type of $(M, r : M \to B\Gamma)$.* ([Ke], see also [PS1])
- **(b.3)** *Let M be a compact oriented Riemannian manifold of dimension $4k+3$, $k > 0$. If $\pi_1(M)$ is not torsion-free, then there are infinitely many manifolds that are homotopy equivalent to M but not homeomorphic to it: they are distinguished by $\rho_{(2)}(D^{sign})$.* (Chang and Wienberger [CW])

Let us come back to the situation of a family. Suppose now there exists a fibration $q : \tilde{M} \to B$ such that $\forall z \in B$ $\tilde{M}_z$ is a normal Γ-covering of M_z, for a fixed group Γ. For example we can obtain this by taking a normal Γ-covering of M. Let $\tilde{\mathcal{D}}$ be the family of Dirac operators lifted on the fibres of the coverings fibration.

Our goal is first to construct a eta form $\hat{\eta}_{(2)}(\tilde{\mathcal{D}}) \in \mathcal{C}^\infty(B, \Lambda T^*B)$ and then to consider the difference $\hat{\eta}_{(2)}(\tilde{\mathcal{D}}) - \hat{\eta}$. The difficulty in the definition of an $\hat{\eta}_{(2)}$ is in the $(t \to \infty)-$asymptotic of the heat operator for the Bismut superconnection on the covering, that is strongly related to the behaviour of the spectrum near zero (see appendix E).

In section 3 we first construct $\hat{\eta}_{(2)}(\tilde{\mathcal{D}})$ under the strong hypothesis that the family on the covering is uniformly invertible

$$\textbf{(\textit{ip}1)} \qquad \exists \mu > 0 \text{ such that } \forall b \in B \quad \begin{cases} \operatorname{spec}(D_b) \cap (-\mu, \mu) = \emptyset \\ \operatorname{spec}(\tilde{D}_b) \cap (-\mu, \mu) = \emptyset \end{cases}$$

(This strong requirement is naturally satisfied for example if the fibres are spin manifolds with metric of positive scalar curvature and if $\mathcal{D}$ and $\tilde{\mathcal{D}}$ are the families of Dirac operators.)

We prove that the difference $\hat{\rho}_{(2)}(\mathcal{D}, R : M \to B\Gamma) := \hat{\eta}(\mathcal{D}) - \hat{\eta}_{(2)}(\tilde{\mathcal{D}})$ is closed. In section 6 we try to answer some questions about the space of metrics of positive scalar curvature along the fibres of a fibration $M \to B$ of spin manifolds. We prove that *$\hat{\rho}(\not{D})$ is constant on the connected components of the space of metrics on the vertical tangent bundle which are positive scalar curvature along the fibres.*

In section 7, following some ideas of [HL], we remove hypothesis (***ip*1**) we study some weaker regularity hypothesis for the spectrum of the family of operators on coverings near

zero. We find out that if the Novikov-Shubin exponent β is greater than $3(\dim B+1)$, and the spectral projections of the family $\tilde{\mathcal{D}}$ are smooth on B, then the rho-form is still well defined and closed. A natural question is then posed: are there fibrations whose family of signature operators on the coverings satisfy these hypothesis on the spectrum? If yes, does it hold that $\hat{\rho}$ is a *fibrewise* homotopy invariant? We state a natural conjecture, on the lines of the results of Keswani [Ke] and Piazza-Schick [PS1].

Acknowledgements This is a part of my researches for the Ph.D thesis. I sincerely wish to thank my advisor Paolo Piazza for having proposed the subject and for the beautiful teachings he has been giving to me in these years.

I also would like to express my gratitude to Moulay Tahar Benameur for interesting discussions, and useful suggestions and remarks.

2 Setting

Let $\pi : M \to B$ be a fibration with fiber a compact manifold F (for the moment suppose $\dim F$ is even). Let $T(M/B)$ be the vertical tangent bundle and let $g_{M/B}$ a metric on it. A connection on TM will be the choice of a subbundle $T^H M \subset TM$ s.t. $T^H M \oplus T(M/B) = TM$. When a connection is given, we will denote with $\mathcal{V}$ and $\mathcal{H}$ the projections on $T^H M$ and $T(M/B)$ relative to the splitting. When X is a section of TB, let X^H denote the unique section of $T^H M$ s.t. $\pi_* X^H = X$.

Let $g_{M/B}$ be a metric on the vertical tangent. Fix also any metric g_B on the base (and lift it on the horizontal bundle). By assuming that $T^H M$ and $T(M/B)$ are orthogonal, TM is endowed with the metric $g = \pi^* g_B \oplus g_{M/B}$. Let ∇^g the Levi-Civita connection on M with respect to the metric $\pi^* g_B \oplus g_{M/B}$. Define

$$\nabla^{M/B} = \mathcal{V} \nabla^g \mathcal{V}$$

on the vertical tangent: $\nabla^{M/B}$ does not depend on g_B (see for example [BGV] proposition 10.2).

Let $E \to M$ be a hermitian vector bundle of Clifford modules on $Cl(T^*M/B, g_{M/B})$, and call $c_m : Cl(T^*M/B, g_{M/B}) \to \mathrm{End}(E_m)$ the Clifford action. Denote $M_b := \pi^{-1}(b)$ and $E_b := E_{|M_b}$. Assume that E is given a connection ∇^E such that

$$[\nabla^E_X, c(\alpha)] = c(\nabla^{M/B}_X \alpha). \tag{4}$$

These data produce a family of Dirac operators $\mathcal{D} = (D_b)_{b \in B}$: explicitely, if $b \in B$ is fixed, let $E_b = E_{|M_b}$ be the restricition of the bundle on the fibre M_b. $\forall m \in M_b$ then $T^*_m M_b = T^*_m(M/B)$, so that restricting to M_b, we have $c_b : Cl(T^*_m M_b) \to \mathrm{End}((E_b))_m$. The composition

$$D_b = c_b \circ \nabla^{E_b} : \mathcal{C}^\infty(M_b, E_b) \to \mathcal{C}^\infty(M_b, E_b)$$

gives the family of operators.

Suppose now there exists a fibration $q : \tilde{M} \to B$ such that $\forall z \in B$, $\tilde{M}_z$ is a normal Γ-covering of M_z, for a fixed group Γ. Let $\tilde{\mathcal{D}}$ be the family of Dirac operators lifted on the fibres of the coverings fibration.

For example we can obtain this by taking a normal Γ-covering of M, as follows from this lemma in appendix F.

Lemma 2.1 — *Let $\pi : M \to B$ be a fibration and let $\tilde{M} \to M$ be a regular Γ-covering of*

the total space M. The composition $q = \pi \circ p : \tilde{M} \to B$ is still a fibration, and the fibre $q^{-1}(b) = \tilde{M}_b$ is a Γ-covering of M_b.

Let $p : \tilde{M} \to M$ be a normal Γ-covering, and let $\tilde{E} := p^*E$ the vector bundle naturally induced on $\tilde{M}$.
It is trivial that $p^*(E)_{|\tilde{M}_b} = (p^*E)_{|\tilde{M}_b} = p_b^*(E_{|M_b}) = \widetilde{E_b}$, where $p_b = p_{|\tilde{M}_b}$. Let $T(\tilde{M}/B) \to \tilde{M}$ be the vertical tangent bundle, which is $T(\tilde{M}/B) = p^*T(M/B)$. The metric $g_{M/B}$ lifts to $g_{\tilde{M}/B}$ on $T(\tilde{M}/B)$. Also the choice of a connection $TM = T(M/B) \oplus T_H M$ gives a splitting of $T\tilde{M}$, posing $\forall \tilde{m} \in \tilde{M}$ $(T_H\tilde{M})_{\tilde{m}} = (dp_{\tilde{m}})^{-1}(T_H M_m) \subseteq T_{\tilde{m}}\tilde{M}$.

$\tilde{E}$ is given the connection $p^*\nabla^E$ and it is trivial also how to define a vertical Clifford action on $\tilde{E}$: the bundle of Clifford algebras $Cl(T^*(\tilde{M}/B), \tilde{g}_{M/B}) \to \tilde{M}$ is clearly the pull back of the Clifford algebras bundle on M. Moreover $\forall \tilde{m} \in \tilde{M}$ $c_{\tilde{m}} : T^*_{\tilde{m}}(\tilde{M}/B) \to \mathrm{End}_{p(\tilde{m})} E_{p(\tilde{m})}$ is exactly $c_{p(\tilde{m})}$ since vector spaces coincide. The family $\tilde{\mathcal{D}} = (\tilde{D}_z)_{z \in B}$ is defined, where $\tilde{D}_z = c \circ \nabla^{\tilde{E}}_{|z}$.

Often in this context one considers also some infinite dimensional fibre bundles associated to the family. Let's recall their definition: $\mathcal{E} \to B$ is the bundle with fibre $\mathcal{E}_b = \mathcal{C}^\infty(M_b, E_b)$, and in parallel $\tilde{\mathcal{E}} \to B$ has fibre $\tilde{\mathcal{E}}_b = \mathcal{C}^\infty(\tilde{M}_b, (p^*E)_b)$. Its spaces of section are given by $\mathcal{C}^\infty(B, \mathcal{E}) = \mathcal{C}^\infty(M, q^*(\Lambda^*B) \otimes E)$ and $\mathcal{C}^\infty(B, \tilde{\mathcal{E}}) = \mathcal{C}^\infty(\tilde{M}, q^*(\Lambda^*B) \otimes \tilde{E})$ respectively. It is clear that $\mathcal{C}^\infty(\tilde{M}, \tilde{E})$ is generated by sections wich are pull-back of sections in $\mathcal{C}^\infty(M, E)$. Then $\mathcal{C}^\infty(B, \tilde{\mathcal{E}})$ is generated by $p^*(\mathcal{C}^\infty(B, \mathcal{E}))$, i.e. $\mathcal{E}$ can be treated as a pull-back of $\tilde{\mathcal{E}}$.

Now we want to construct a superconnection $\tilde{\mathbb{B}}$ adapted to the family $\tilde{\mathcal{D}}$. It will be a differential operator

$$\tilde{\mathbb{B}} : C^\infty(\tilde{M}, \tilde{E} \otimes \pi^*\Lambda T^*B) \to C^\infty(\tilde{M}, \tilde{E} \otimes \pi^*\Lambda T^*B)$$

which satisfies conditions in definition A.19.

Moreover we want that the $t \to 0$-asymptotic of $e^{-t\tilde{\mathbb{B}}}$ has the usual properties of the Bismut superconnection.

Bismut superconnection for $\tilde{\mathcal{D}}$. To construct a Bismut superconnection $\tilde{\mathbb{B}}$ adapted to the family $\tilde{\mathcal{D}}$ we can either pull-back the Bismut superconnection adapted to $\mathcal{D}$, or we can repeat the geometric construction in A.2: both methods give the same result $\tilde{\mathbb{B}}$.

Pull-back construction: consider a family as described before: in general if $\mathbb{A}$ is a superconnession adapted to the family $\mathcal{D}$, then one can define $\tilde{\mathbb{A}}$ adapted to $\tilde{\mathcal{D}}$: posing $\forall f \in \mathcal{C}^\infty(\tilde{M})$, $\forall s \in \mathcal{C}^\infty(M, E)$

$$\tilde{\mathbb{A}}(f \cdot p^*s) := d_B \wedge f p^*s + f p^*(\mathbb{A}s)$$

and then $\forall \beta \in \mathcal{A}(B)$, $\forall s \in \mathcal{C}^\infty(\tilde{M}, \tilde{E})$

$$\tilde{\mathbb{A}}(q^*\beta \otimes s) = q^*d_B\beta + (-1)^{|\beta|} q^*\beta \wedge \tilde{\mathbb{A}}s$$

By construction $\tilde{\mathbb{A}}$ is odd w.r.t. the total grading, Leibniz rule holds, and the superconnection is adapted to $\tilde{\mathcal{D}}$ in fact $\forall f \in \mathcal{C}^\infty(\tilde{M})$, $\forall s \in \mathcal{C}^\infty(M, E)$

$$\tilde{\mathbb{A}}_{[0]}(f \cdot p^*s) = f p^*(\mathbb{A}_{[0]}s) = f p^*(\mathcal{D}s) = \tilde{\mathcal{D}}(f \cdot p^*s)$$

Geometric construction. We can repeat step by step the construction of the Bismut superconnection (see appendix A.2). Let $\nabla^{\tilde{M}/B}$ on $T(\tilde{M}/B)$ the canonical connection on the vertical bundle constructed (see [BGV], proposition 10.2), $\nabla^{\tilde{M}/B} := \tilde{P}\nabla^{\tilde{g}}\tilde{P}$ where $\nabla^{\tilde{g}}$ is the Levi-Civita connection on $T\tilde{M}$ w.r.t. the metric $\tilde{g} = g_{\tilde{M}/B} \oplus g_B$. It is clear that $\nabla^{\tilde{g}} = p^*\nabla^g$. All tensors recalled in page 26 can be written also for the fibration $\tilde{M} \to B$ and the family $\tilde{\mathcal{D}}$: denote them with the letters $\tilde{k}, \tilde{\Omega}, \tilde{T}$.

Repeating the construction we obtain the superconnection $\tilde{\mathbb{B}}$

$$\tilde{\mathbb{B}} = \sum_i c(e^i)\nabla^{\tilde{E}}_{e_i} + \sum_\alpha dy^\alpha \wedge \left(\nabla_{f_\alpha} + \frac{1}{2}\tilde{k}(f_\alpha)\right) - \frac{1}{4}\sum_{\alpha<\beta} c(\tilde{\Omega}(f_\alpha, f_\beta))dy_\alpha \wedge dy_\beta \tag{5}$$

which is exactly the pull back of the Bismut superconnection $\tilde{\mathbb{B}}$.

3 Analysis for the family $\tilde{\mathcal{D}}$ and the Bismut superconnection

In our setting we are dealing with two families of Dirac operators: $\mathcal{D}$ and $\tilde{\mathcal{D}}$. Let $\tilde{\mathbb{B}}$ be the Bismut superconnection adapted to $\tilde{\mathcal{D}}$ constructed as above. Then $\tilde{\mathbb{B}} = \tilde{\mathcal{D}} + \tilde{\mathbb{B}}_{[1]} + \tilde{\mathbb{B}}_{[2]}$, where $\tilde{\mathbb{B}}_{[i]} : \mathcal{A}^\bullet(B, \tilde{E}) \to \mathcal{A}^{\bullet+i}(B, \tilde{E})$, $i = 1, 2$.

The square $\tilde{\mathbb{B}}^2$ is a family of vertical operators and from its explicit construction (see for example [BGV], theorem 10.17) we know that

$$\tilde{\mathbb{B}}^2 = \tilde{\mathcal{D}}^2 + \tilde{\mathcal{F}}_{[+]}$$

where $\tilde{\mathcal{D}}^2$ is a family of generalized Laplacians, and $\tilde{\mathcal{F}}_{[+]}$ is a family of zero-order operators of with coefficients in $\Lambda^* T^* B$. In other words for any fixed $z \in B$ we have $\tilde{F}^z = (\tilde{D}^z)^2 + \tilde{F}^z_{[+]}$, where $\tilde{F}^z_{[+]}$ is the lift of the corresponding zero order differential operator on M_z.

Remark — Since $\tilde{M}_z$ is the covering of M_z, the bundle $\tilde{E}_z \to \tilde{M}_z$ is bounded geometry. This implies that $\tilde{F}^z_{[+]}$ belongs to the space of bounded C^∞-sections $UC^\infty(\tilde{M}_z, \mathrm{End}(\tilde{E}_z) \otimes \bigwedge T^*_z B)$. (See appendix C.1 for definitions).

3.1 Construction of the heat kernel for $\tilde{\mathbb{B}}^2$

The main property of the Bismut superconnection $\mathbb{B}$ is the behavior of the $t \to 0$-asymptotic of $e^{-\mathbb{B}^2_t}$. In the direction to obtain the corresponding property for $\tilde{\mathbb{B}}$, we first give a precise meaning to the quantity $e^{-t\tilde{\mathbb{B}}^2}$, where $\tilde{\mathbb{B}}$ is a superconnection adapted to a family of operators on the coverings. It is in fact the family $(e^{-t\tilde{F}^z})_{z\in B}$. Fixed $z \in B$ we want to construct the heat kernel for the operator $\tilde{F}^z = (\tilde{D}^z)^2 + \tilde{F}^z_{[+]}$. Observe that $(\tilde{D}_z)^2$ has a heat operator $e^{-t\tilde{D}^2_z}$ (by functional calculus). With the Volterra series we write formally

$$e^{-t\tilde{F}^z} = e^{-t\tilde{D}^2_z} + \sum_{k>0} (-t)^k I^z_k \tag{6}$$

with

$$I^z_k = \int_{\triangle_k} e^{-\sigma_0 t\tilde{D}^2_z} \tilde{F}^z_{[+]} e^{-\sigma_1 t\tilde{D}^2_z} \dots \tilde{F}^z_{[+]} e^{-\sigma_k t\tilde{D}^2_z} d\sigma_1 \dots d\sigma_k \quad , \sum_{i=0}^{i=k} \sigma_i = 1$$

As in chapter 9 of [BGV] we observe that the sum in (6) is finite, since the composition of k times the operator $\tilde{\mathcal{F}}_{[+]}$ gives an operator which increases of k the degree in the exterior algebra $\Lambda T^*_z B$, and $\Lambda^k T^* B = 0$ for $k > \dim B$. Each term $e^{-\sigma_0 t\tilde{\mathcal{D}}^2} \tilde{\mathcal{F}}_{[+]} e^{-\sigma_1 t\tilde{\mathcal{D}}^2} \dots \tilde{\mathcal{F}}_{[+]} e^{-\sigma_k t\tilde{\mathcal{D}}^2}{}_{|z}$ makes sense and is a smoothing operator: in fact since the condition $\sum \sigma_i = 1$ implies that at least one of the σ_i is not zero, the corresponding $e^{-\sigma_i t\tilde{D}^2_z}$ is smoothing, and each operator $\tilde{F^z}_{[+]}$ is a bounded zero order differential operator on $L^2(\tilde{M}_z, \tilde{E}_z \otimes \bigwedge T^*_z B)$. Moreover the dependence on

σ is continuos and each I_k turns out to be a smoothing operator. For each $z \in B$ fixed, we have constructed $e^{-t\tilde{F}^z} \in \mathrm{Op}^{-\infty}(\tilde{M}_z, \tilde{E}_z) \otimes \Lambda T_z^* B$. Observe that $\tilde{\mathbb{B}}\gamma = \gamma\tilde{\mathbb{B}}$ and, as consequence, that $\forall z\ e^{-t\tilde{F}^z} \in \mathrm{Op}_\Gamma^{-\infty}(\tilde{M}_z, \tilde{E}_z) \otimes \pi^* \Lambda T_z^* B$. Hence by lemma D.33 in appendix C, $e^{-t\tilde{F}^z}$ is Γ-trace class. Let $\tilde{p}_t^z(x,y)$ is the heat kernel of $\tilde{F}^z$, then as usual one proves that $\tilde{p}_t^z(x,y)$ is smooth in $z \in B$.

DEFINITION 3.2 — Define $\mathcal{K}_\Gamma(\tilde{E}) \to B$ as the infinite dimensional bundle whose fibre is $\mathcal{K}_\Gamma(\tilde{E})_z = \{K \in \Psi^{-\infty}(\tilde{M}_z, \tilde{E}_z) \ : \ K \cdot \gamma = \gamma \cdot K \ \ \forall \gamma \in \Gamma\}$.

Then $e^{-t\tilde{\mathcal{F}}}$ is a section of $\mathcal{K}_\Gamma(\tilde{E}) \otimes \Lambda T^* B$.

Notation from now on the Schwartz kernel of an operator T will be denoted by $[T](x,y)$. We will need the following basic fact, proved in [Do] and [Do2]:

LEMMA 3.3 — *For $t < T_0$*

$$\left|[e^{-t\tilde{\mathcal{F}}}](\tilde{x},\tilde{y})\right| \leq c_1 t^{-\frac{n}{2}} e^{-c_2 \frac{d^2(\tilde{x},\tilde{y})}{s}} \tag{7}$$

Consider now the *rescaled superconnection* $\tilde{\mathbb{B}}_s := s^{\frac{1}{2}} \delta_s \circ \tilde{\mathbb{B}} \delta_s^{-1}$ where δ_s is the operator which, restricted to $C^\infty(\tilde{M}_z, \tilde{E}_z \otimes \bigwedge^k T_z^* B)$ is multiplication by $s^{-\frac{k}{2}}$. In our case $\tilde{\mathbb{B}}_s = s^{\frac{1}{2}} \tilde{\mathbb{B}}_{[0]} + \tilde{\mathbb{B}}_{[1]} + s^{-\frac{1}{2}} \tilde{\mathbb{B}}_{[2]}$. We have

$$\tilde{\mathbb{B}}^2_{s|z} = s\delta_s \circ \tilde{\mathcal{F}} \circ \delta_s^{-1}|_z = s\ \delta_s(\tilde{F})_{|z} = s\ \delta_s(\tilde{D}_z^2 + \tilde{F}^z_{[+]})\delta_s^{-1} = s(\tilde{D}_z^2 + \delta_s(\tilde{F}^z_{[+]})) \tag{8}$$

Now, from lemma D.33 in appendix, $e^{-\tilde{\mathbb{B}}^2_s}{}_{|z} = e^{-s(\tilde{D}_z^2 + \delta_s(\tilde{F}^z_{[+]}))}$ belongs to $\mathcal{B}^1_\Gamma(L^2(\tilde{M}_z, \tilde{E}_z)) \otimes \Lambda T_z^* B$. We can write

$$\mathrm{Str}_\Gamma e^{-\tilde{\mathbb{B}}^2_s}{}_{|z} = \mathrm{Str}_\Gamma \left(e^{-s(\tilde{D}_z^2 + \delta_s(\tilde{F}^z_{[+]}))}\right) = \int_{\mathcal{I}_z} \mathrm{Str}_{\tilde{E}_x} \left[e^{-s(\tilde{D}_z^2 + \delta_s(\tilde{F}^z_{[+]}))}\right](x,x) d\,\mathrm{vol}_{\tilde{M}_z}(x)$$

where $\mathcal{I}_z$ is a fundamental domain in $\tilde{M}_z$.
Its Γ-supertrace is an element of $\Lambda T_z^* B$ hence $\mathrm{Str}_\Gamma(e^{-\tilde{\mathbb{B}}^2_s}) \in \mathcal{C}^\infty(B, \Lambda T^* B)$. The following lemma is simple:

LEMMA 3.4 —

$$\frac{d}{ds}\left(\mathrm{Str}_\Gamma(e^{-\tilde{\mathbb{B}}^2_s})\right) = -d\,\mathrm{Str}_\Gamma\left(\frac{d\tilde{\mathbb{B}}_s}{ds} e^{-\tilde{\mathbb{B}}^2_s}\right)$$

PROOF— $\displaystyle \frac{d}{ds}\left(\mathrm{Str}_\Gamma(e^{-\tilde{\mathbb{B}}^2_s})\right)_{|z} = \frac{d}{ds}\mathrm{Str}_\Gamma\left(e^{-s(\tilde{D}_z^2 + \delta_s(\tilde{F}^z_{[+]}))}\right) =$

$$= \frac{d}{ds}\mathrm{Str}\left(\chi_{\mathcal{I}} e^{-s(\tilde{D}_z^2 + \delta_s(\tilde{F}^z_{[+]}))}\chi_{\mathcal{I}}\right)$$

where $\mathcal{I}$ is a fundamental domain in $\tilde{M}_z$ and where the last equality follows from proposition D.32 in appendix; by the classical equality (see for example [BGV] prop. 1.41) this equals

$$= -d\,\mathrm{Str}(\chi_{\mathcal{I}}(\frac{d\tilde{\mathbb{B}}_s}{ds} e^{-\tilde{\mathbb{B}}_s})\chi_{\mathcal{I}}) = -d\,\mathrm{Str}_\Gamma(\frac{d\tilde{\mathbb{B}}_s}{ds} e^{-\tilde{\mathbb{B}}_s})$$

□

Integrating on the interval (t,T) one gets as usual a **transgression formula**

$$\mathrm{Str}_{\Gamma}\left(e^{-\tilde{\mathbb{B}}_T^2}\right) - \mathrm{Str}_{\Gamma}\left(e^{-\tilde{\mathbb{B}}_t^2}\right) = -d\int_t^T \mathrm{Str}_{\Gamma}\left(\frac{d\tilde{\mathbb{B}}_s}{ds}e^{-\tilde{\mathbb{B}}_s^2}\right)ds \tag{9}$$

4 The $t\to 0$ asymptotic

In this section we analyse the $t\to 0$ asymptotic for $e^{-\tilde{\mathbb{B}}_t^2}$. Here we don't need any regularity condition on the spectrum of $\mathcal{D}$ and $\tilde{\mathcal{D}}$. We'll prove the following propositions:

PROPOSITION 4.5 — *If $\tilde{\mathbb{B}}$ is the Bismut superconnection, then*

$$\lim_{t\to 0}\mathrm{Str}_{\Gamma}\left(e^{-\mathbb{B}_t^2}\right) = \int_{M/B}\hat{A}(M/B)\,\mathrm{ch}\,E/S \tag{10}$$

PROPOSITION 4.6 — *The term* $\mathrm{Str}_{\Gamma}\dfrac{d\tilde{\mathbb{B}}_t}{dt}e^{-\tilde{\mathbb{B}}_t^2}{}_{|z}$ *is integrable for $t\to 0$.*

To prove these propositions we show, following the idea of Atiyah [A1] pag.416, that $e^{-t\mathcal{F}}$ and $e^{-t\tilde{\mathcal{F}}}$ have the same asymptotic expansion for $t\to 0$. As observed by Atiyah, the first step is the following lemma

LEMMA 4.7 — *If $F(t,\tilde{x},\tilde{y}) := \sum_{g\in\Gamma}[e^{-t\tilde{\mathcal{F}}}](\tilde{x},\tilde{y}g)$, then*

$$F(t,\tilde{x},\tilde{y}) = [e^{-t\mathcal{F}}](p(\tilde{x}),p(\tilde{y})) \tag{11}$$

PROOF —

$$\left(\frac{\partial}{\partial t}+\tilde{\mathcal{F}}_{\tilde{x}}\right)F(t,\tilde{x},\tilde{y}) = \left(\frac{\partial}{\partial t}+\tilde{\mathcal{F}}_{\tilde{x}}\right)\left(\sum_{g\in\Gamma}[e^{-t\tilde{\mathcal{F}}}](\tilde{x},\tilde{y}g)\right) = 0$$

Moreover $\lim_{t\to 0}\int_{\tilde{M}}F(t,\tilde{x},\tilde{y})s(\tilde{y})d\tilde{y} = s(\tilde{x})$. Now observe that $F(t,\tilde{x},\tilde{y})$ is $\Gamma\times\Gamma-$invariant, in fact

$$F(t,\tilde{x}\cdot g_1,\tilde{y}\cdot g_2) = \sum_{g\in\Gamma}[e^{-t\tilde{\mathcal{F}}}](\tilde{x}g_1,\tilde{y}gg_2) =$$

$$\sum_{g\in\Gamma}[e^{-t\tilde{\mathcal{F}}}](\tilde{x},\tilde{y}gg_2(g_1)^{-1}) = F(t,\tilde{x},\tilde{y}).$$

where we have used that $\tilde{\mathcal{F}}$ is the lift of $\mathcal{F}$ so that $[e^{t\tilde{\mathcal{F}}}]$ is Γ-invariant w.r.t.the diagonal action.

$\Gamma\times\Gamma$-invariance implies $F=\pi^*G$ for some kernel G on M. Now

$$0 = \left(\frac{\partial}{\partial t}+\tilde{\mathcal{F}}_{\tilde{x}}\right)F(t,\tilde{x},\tilde{y}) = \left(\frac{\partial}{\partial t}+\tilde{\mathcal{F}}_{\tilde{x}}\right)\pi^*G(t,\tilde{x},\tilde{y}) = \pi^*\left(\frac{\partial}{\partial t}+\mathcal{F}_{\tilde{x}}\right)G(t,x,y)$$

then $(\frac{\partial}{\partial t}+\mathcal{F}_{\tilde{x}})G(t,x,y) = 0$ and also $\lim_{t\to 0}G(t) = I$ since the fundamental solution of the heat equation is unique, then

$$G(t,\tilde{x},\tilde{y}) = [e^{-t\tilde{\mathcal{F}}}](x,y)$$

□

From this lemma it follows that

$$[e^{-t\mathcal{F}}](p(\tilde{x}),p(\tilde{y}))-[e^{-t\tilde{\mathcal{F}}}](\tilde{x},\tilde{y})=$$
$$=\sum_{g\in\Gamma}[e^{-t\tilde{\mathcal{F}}}](\tilde{x},\tilde{y}g)-[e^{-t\tilde{\mathcal{F}}}](\tilde{x},\tilde{y})=\sum_{g\neq e}[e^{-t\tilde{\mathcal{F}}}](\tilde{x},\tilde{y}g)$$

On the diagonal

$$[e^{-t\mathcal{F}}](p(\tilde{x}),p(\tilde{x}))-[e^{t\tilde{\mathcal{F}}}](\tilde{x},\tilde{x})=\sum_{g\neq e}[e^{-t\tilde{\mathcal{F}}}](\tilde{x},\tilde{x}g) \tag{12}$$

Now we evaluate $\left|\sum_{g\neq e}[e^{-t\tilde{\mathcal{F}}}](\tilde{x},\tilde{x}g)\right|$: fix $\Lambda>0$

$$\left|\sum_{g\neq e}[e^{-t\tilde{\mathcal{F}}}](\tilde{x},\tilde{x}g)\right|\leq\left|\sum_{\substack{g\neq e\\ d(\tilde{x},\tilde{x}g)>\Lambda}}[e^{-t\tilde{\mathcal{F}}}](\tilde{x},\tilde{x}g)\right|+\left|\sum_{\substack{g\neq e\\ d(\tilde{x},\tilde{x}g)\leq\Lambda}}[e^{-t\tilde{\mathcal{F}}}](\tilde{x},\tilde{x}g)\right|$$

To estimate the term corresponding to $d(\tilde{x},\tilde{x}g)>\Lambda$: let $N(r):=\#\{\{\tilde{x}g,\ g\in\Gamma\}\cap B(\tilde{x},r)\}$. If the sectional curvature of the compact manifold M_z is bounded below by $-K^2$, then the following estimate holds[1], we

$$N(r)\leq cost\cdot e^{(n-1)Kr} \tag{13}$$

Then

$$\left|\sum_{\substack{g\neq e\\ d(\tilde{x},\tilde{x}g)>\Lambda}}[e^{-t\tilde{\mathcal{F}}}](\tilde{x},\tilde{x}g)\right|\leq\sum_{\substack{g\neq e\\ d(\tilde{x},\tilde{x}g)>\Lambda}}\left|[e^{-t\tilde{\mathcal{F}}}](\tilde{x},\tilde{x}g)\right|\leq$$
$$\leq\int_{\Lambda}^{\infty}cost\cdot s^{-\frac{n}{2}}e^{-\frac{R^2}{s}}e^{(n-1)KR}dR=cost\cdot s^{-\frac{n}{2}}\int_0^{\infty}e^{-\frac{(\Lambda+x)^2}{s}}e^{(n-1)K(\Lambda+x)}dx=$$
$$=C\cdot s^{-\frac{n}{2}}\int_0^{\infty}e^{-\frac{\Lambda^2+x^2+2x\Lambda}{s}}e^{(n-1)K(\Lambda+x)}dx=$$
$$=C\cdot s^{-\frac{n}{2}}e^{-\frac{\Lambda^2}{s}}e^{(n-1)K\Lambda}\int_0^{\infty}e^{-\frac{x^2}{s}-x(\frac{2\Lambda}{s}+(1-n)K)}dx\leq$$
$$\leq C\cdot s^{-\frac{n}{2}}e^{-\frac{\Lambda^2}{s}}e^{(n-1)K\Lambda}\int_0^{\infty}e^{-x(\frac{2\Lambda}{s}-(n-1)K)}dx=C\cdot s^{-\frac{n}{2}}e^{-\frac{\Lambda^2}{s}}e^{(n-1)K\Lambda}\frac{1}{\frac{2\Lambda}{s}-(n-1)K}$$

For sufficiently big Λ this is $o(e^{-\frac{A\delta^2}{s}})$ for $s\to 0$.
The remaining term $\left|\sum_{\substack{g\neq e\\ d(\tilde{x},\tilde{x}g)\leq\Lambda}}[e^{-t\tilde{\mathcal{F}}}](\tilde{x},\tilde{x}g)\right|$ is a finite sum. Let δ be the length of the shortest closed geodesic on M which is in a nontrivial homotopy class. Each term is then bounded by $cost\cdot s^{-\frac{n}{2}}e^{-\frac{\delta^2}{s}}$, so that the term is $o(e^{-\frac{A\delta^2}{s}})$. Finally

$$\left|[e^{-t\mathcal{F}}](p(\tilde{x}),p(\tilde{x}))-[e^{t\tilde{\mathcal{F}}}](\tilde{x},\tilde{x})\right|=o(e^{-\frac{A\delta^2}{t}})$$

[1]Let ϵ small enaugh s.t. $B(\tilde{x},\epsilon)\cap B(\tilde{x}g)=\emptyset$ $g\neq\epsilon$. Then $\mathrm{vol}(B(\tilde{x},r))\geq N(r)\,\mathrm{vol}(B(\tilde{x},\epsilon))$ so that

$$N(r)\leq\frac{\mathrm{vol}(B(\tilde{x},r))}{\mathrm{vol}(B(\tilde{x},\epsilon))}$$

Now, using Bishop inequality and the fact that the ball of radius r in a space of constant sectional curvature $-K^2$ is $V^{-K^2}(r)\sim ce^{-(n-1)r}$, the estimate follows. (see [Mi])

hence the two kernels have the same asymptotic for $t \to 0$. Explicitly,

$$[e^{-t\mathcal{F}}](x,x) \sim (4\pi t)^{-\frac{n}{2}} \sum_{i=0}^{\infty} t^i k_i(x)$$

$$[e^{-t\tilde{\mathcal{F}}}](\tilde{x},\tilde{x}) \sim (4\pi t)^{-\frac{n}{2}} \sum_{i=0}^{\infty} t^i \tilde{k}_i(\tilde{x})$$

with $\tilde{k}_i(\tilde{x}) = k_i(p(\tilde{x}))$.

PROOF— (of proposition 4.5)

$$\lim_{s\to 0} \operatorname{Str}_{\Gamma}(e^{-\tilde{\mathcal{F}}_s}) = \lim_{s\to 0} \int_{\mathcal{I}} \delta_s \left(\operatorname{Str}_{\tilde{E}_{\tilde{x}}}[e^{-s\tilde{\mathcal{F}}}](\tilde{x},\tilde{x})\right) d\tilde{x}$$

the integrand

$$\lim_{s\to 0} \delta_s \left(\operatorname{Str}_{\tilde{E}_{\tilde{x}}}[e^{-s\tilde{\mathcal{F}}}](\tilde{x},\tilde{x})\right) = \lim_{s\to} \delta_t \left(\operatorname{Str}_{E_x}[e^{-t\mathcal{F}}](p(\tilde{x}),p(\tilde{x}))\right) =$$

$$= (2\pi i)^{-\frac{n}{2}} \hat{A}(M/B)\operatorname{ch}(E/S).$$

□

PROOF— (of proposition 4.6) To prove that the term $\beta(s) = \operatorname{Str}_{\Gamma}(\frac{d\tilde{\mathbb{B}}_t}{dt} e^{-\tilde{\mathbb{B}}_t^2})$ is integrable for $s \to 0$ we follow [BGV] chapter 10, pag. 340: consider the rescaled superconnection $\tilde{\mathbb{B}}_s$ as a one-parameter family of superconnections, $s \in \mathbb{R}^+$. Consider a new family $\breve{M} = \tilde{M} \times \mathbb{R}^+ \to B \times \mathbb{R}^+ =: \breve{B}$. Let $\breve{E} = \tilde{E} \times \mathbb{R}^+$. The computations in [BGV] give that the Bismut superconnection for the new family is $\breve{\mathbb{B}} = \tilde{\mathbb{B}}_s + d_{\mathbb{R}^+} - \frac{n}{4s} ds$. The rescaling of $\breve{\mathbb{B}}$ gives

$$\breve{\mathbb{B}}_t = \tilde{\mathbb{B}}_{st} + d_{\mathbb{R}^+} - \frac{n}{4s} ds$$

and

$$\breve{\mathcal{F}}_t = \tilde{\mathbb{B}}_{st}^2 + t \frac{d\tilde{\mathbb{B}}_s}{ds} \wedge ds$$

so that

$$e^{-\breve{\mathcal{F}}_t} = e^{-\tilde{\mathcal{F}}_{st}} - \int_0^1 e^{-u\tilde{\mathcal{F}}_{st}} t \frac{d\tilde{\mathbb{B}}_s}{ds} e^{-(1-u)\tilde{\mathcal{F}}_{st}} \wedge ds = e^{-\tilde{\mathcal{F}}_{st}} - \frac{\partial \tilde{\mathbb{B}}_{st}}{\partial s} e^{-\tilde{\mathcal{F}}_{st}} \wedge ds.$$

Then

$$\operatorname{Str}_{\Gamma}\left(e^{-\breve{\mathcal{F}}_t}\right) = \operatorname{Str}_{\Gamma}(e^{-\tilde{\mathcal{F}}_{st}}) - \operatorname{Str}_{\Gamma}\left(\frac{\partial \tilde{\mathbb{B}}_{st}}{\partial s} e^{-\tilde{\mathcal{F}}_{st}}\right) ds$$

Since $\breve{\mathcal{F}}$ is the curvature of a Bismut superconnection, then the asymptotic expansion of $\operatorname{Str}_{\Gamma}(e^{-\breve{\mathcal{F}}_t})$ does not have singular terms

$$\operatorname{Str}_{\Gamma}(e^{-\breve{\mathcal{F}}_t}) \sim \sum_{j=0}^{\infty} t^{\frac{j}{2}} (\Phi_{\frac{j}{2}} - \alpha_{\frac{j}{2}} ds)$$

Computing in $s=1$, since $\frac{\partial\tilde{\mathbb{B}}_{st}}{\partial s}=t\frac{\partial\tilde{\mathbb{B}}_s}{\partial s}$

$$\mathrm{Str}_\Gamma\left(t\frac{\partial\tilde{\mathbb{B}}_s}{\partial s}e^{-\tilde{\mathcal{F}}_t}\right)\sim\sum_{j=0}^{\infty}t^{\frac{j}{2}}\alpha_{\frac{j}{2}}$$

and

$$\mathrm{Str}_\Gamma\left(\frac{\partial\tilde{\mathbb{B}}_s}{\partial s}e^{-\tilde{\mathcal{F}}_t}\right)\sim\sum_{j=0}^{\infty}t^{\frac{j}{2}-1}\alpha_{\frac{j}{2}}$$

Let's compute α_0: from the local formula

$$\Phi_0-\alpha_0 ds=\lim_{t\to 0}\mathrm{Str}_\Gamma\left(e^{-\check{\mathcal{F}}_t}\right)=(2\pi i)^{-\frac{n}{2}}\int_{\check{M}/\check{B}}\hat{A}(\check{M}/\check{B}) \tag{14}$$

since $\check{M}_{(z,s)}=\tilde{M}_z\times\{s\}$and the differential forms are pulled back from those on $\tilde{M}\to B$, then the right hand side of (14) does not contain ds so that $\alpha_0=0$. This implies that

$$\mathrm{Str}_\Gamma(\frac{d\tilde{\mathbb{B}}_t}{dt}e^{-\tilde{\mathbb{B}}_t^2})\sim\sum_{j=1}^{\infty}t^{\frac{j}{2}-1}\alpha_{\frac{j}{2}}$$

□

5 The $t\to\infty$ asymptotic under hypothesis (ip1)

In this section we will work under this strong hypothesis:

(*ip*1) $\quad\exists\mu>0$ such that $\forall b\in B\quad\begin{cases}\mathrm{spec}(D_b)\cap(-\mu,\mu)=\emptyset\\ \mathrm{spec}(\tilde{D}_b)\cap(-\mu,\mu)=\emptyset\end{cases}$

(later we will remove this assumption). In this case it is not difficult to prove that the asymptotic behaviour of $e^{-\tilde{\mathbb{B}}_t}$ for $t\to\infty$ makes no obstructions in the direction of defining $\hat{\eta}_{(2)}$. In fact we prove the following two lemmas.

LEMMA 5.8 — *Under the hypothesis **(ip 1)** one has*

$$\lim_{t\to\infty}\mathrm{Str}_\Gamma(e^{-\tilde{\mathbb{B}}_t^2})=0$$

LEMMA 5.9 — *Under the hypothesis **(ip 1)***

$$\beta_z(t):=\mathrm{Str}_\Gamma\left(\frac{d\tilde{\mathbb{B}}_t}{dt}e^{-\tilde{\mathbb{B}}^2_t}{}_{|z}\right)=\mathcal{O}(t^{-\delta})\quad\forall\delta>0$$

so that $\beta_z(t)$ is integrable for $s\to\infty$.

PROOF— (of lemmas 5.8 and 5.9) Let $\mathcal{M}=\mathbb{C}^\infty(B,\Lambda T^*B\otimes\mathrm{End}\,\tilde{\mathcal{E}})$ the big set which contains in particular all families of differential operators with differential forms coefficients: it is filtered by $\mathcal{M}_i=$ sections of $\sum_{j\geq i}\Lambda^jT^*B\otimes\mathrm{End}\,\tilde{\mathcal{E}}$. In the same way, $\mathcal{K}(\tilde{\mathcal{E}})\otimes\Lambda T^*B\to B$ is the infinite dimensional bundle with fibre $\mathcal{K}(\tilde{\mathcal{E}})_z\otimes\Lambda T^*_zB$ equals the algebra of smoothing

operators on $\tilde{E}$ with differential forms coefficients. The space $\mathcal{N}$ of sections of the infinite dimensional bundle $\mathcal{K}(\tilde{\mathcal{E}})\otimes \to B$ is filtered by $\mathcal{N}_i =$ sections of $\sum_{j\geq i} \Lambda^j T^*B \otimes \mathcal{K}(\tilde{\mathcal{E}})$.

We make the following remark, used in [HL] (here we use that the manifold $\tilde{M}_z$ and the bundle $\tilde{E}_z \to \tilde{M}_z$ are of bounded geometry): if $T \in \mathcal{N}$, then $\forall z \in B \quad \mathcal{N}_z \in \mathrm{Op}^{-\infty}(\tilde{M}_z, \tilde{E}_z)$, and by proposition [C.30] in appendix, its Schwartz kernel $[\mathcal{N}_z]$ satisfies that for sufficiently large l it holds that $\exists c_l^z$ s.t.

$$\forall x, y \in \tilde{M} \quad | [\mathcal{N}_z](x,y) | \leq c_l^z \, \|\mathcal{N}_z\|_{-l,l} \tag{15}$$

Then, for Γ-trace class operators, an estimate of $\|\mathcal{N}_z\|_{-l,l}$ produces an estimate of $\mathrm{Str}_\Gamma(\mathcal{N})$.

From (8) we have $\mathbb{B}^2_{s|z} = s(\tilde{D}_z^2 + \delta_s(\tilde{F}^z_{[+]}))$. To simplify notations, we consider now $z \in B$ fixed and we write $\tilde{D}$ for $\tilde{D}_z$. Also denote $\delta_t(\tilde{F}^{[+]}) := C_t$ so that our heat kernel is $e^{-t(\tilde{D}+C_t)} = e^{-t\tilde{D}} + \sum_{k>0}(-t)^k \int_{\triangle_q} \mathcal{A}_q$. The assumption (**ip1**) implies that the inverse G of $\tilde{D}^2$ satisfies $\|G\|_{s,s} \leq \dfrac{1}{\mu} \quad \forall s \in \mathbb{R}$. This implies, with estimates as in [Lo] (pag. 215, proof of proposition 25), that $\exists A$

$$\left|[e^{-t\tilde{D}^2}](x,y)\right| \leq e^{-At} \quad \text{for large } T \tag{16}$$

Then our goal is now to estimate $|| \sum_{q>0}(-t)^k \int_{\triangle_q} \mathcal{A}_q||_{-l,l}$. Fix q and consider the integrand $\mathcal{A}_q = e^{-\sigma_0 t\tilde{D}^2} C_t e^{-\sigma_1 t\tilde{D}^2} \dots \tilde{C}_t e^{-\sigma_q t\tilde{D}^2}$. There exists i s.t. $\sigma_i \geq \frac{1}{q+1}$. Let for example $i = q$. $\tilde{C}_t$ is of order zero and belongs to $\mathcal{M}_\infty$. Then $t^{-\frac{1}{2}}\tilde{C}_t$ has coefficients which are uniformly bounded in t, for $t > \frac{1}{2}$, so that $||t^{-\frac{1}{2}}\tilde{C}_t||_{l,l} \leq c_1$. Write each term $\mathcal{A}_q = e^{-\sigma_0 t\tilde{D}^2} t^{-\frac{1}{2}} C_t e^{-\sigma_1 t\tilde{D}^2} \dots t^{-\frac{1}{2}} \tilde{C}_t e^{-\sigma_q t\tilde{D}^2} t^{\frac{q}{2}}$

$\forall \sigma \neq \sigma_q$ we have $\left\|e^{-t\sigma\tilde{D}^2} t^{-\frac{1}{2}}\tilde{C}_t\right\|_{l,l} \leq \left\|e^{-t\sigma\tilde{D}^2}\right\|_{l,l} ||t^{-\frac{1}{2}}\tilde{C}_t||_{l,l} \leq c_1$. On the other hand

$$\left\|e^{-t\sigma_q\tilde{D}^2}\right\|_{-l,l} = \left\|t^{-m} G^m (t\tilde{D}^2)^m e^{-t\sigma_q\tilde{D}^2}\right\|_{-l,l} \leq t^{-m} \|G^m\|_{l,l} \left\|(t\tilde{D}^2)^m e^{-t\sigma_q\tilde{D}^2}\right\|_{-l,l} \leq$$

$$\leq t^{-m} \frac{1}{\mu^m} \sup_z \{(tz)^m (1+z^2)^l e^{-\frac{t}{q+1} z^2}\} \leq ct^{-m}$$

Hence using also (16)

$$\left|[e^{-t(\tilde{D}^2+C_t)}](x,y)\right| = \left|[e^{-t\tilde{D}^2}](x,y) + [\sum \int \mathcal{A}_q](x,y)\right| = \mathcal{O}(t^{-\delta}) \quad \forall \delta > 0$$

which proves Lemma 5.8. Now to prove Lemma 5.9 write

$$\frac{d\tilde{\mathbb{B}}_s}{ds} = \frac{d}{ds}\left(s^{\frac{1}{2}}\tilde{\mathbb{B}}_{[0]} + \tilde{\mathbb{B}}_{[1]} + s^{-\frac{1}{2}}\tilde{\mathbb{B}}_{[2]}\right) = \frac{1}{2}s^{-\frac{1}{2}}\left(\tilde{\mathbb{B}}_{[0]} - s^{-1}\tilde{\mathbb{B}}_{[2]}\right) =$$

$$= \frac{1}{2}s^{-\frac{1}{2}}\left(\tilde{\mathcal{D}} - s^{-1}\tilde{\mathbb{B}}_{[2]}\right)$$

which shows that $\dfrac{d\tilde{\mathbb{B}}_s}{ds}$ is a family of vertical operators.

We write for each fixed $z \in B$, $\frac{d\tilde{\mathbb{B}}_s}{ds} = \frac{1}{2}s^{-\frac{1}{2}}\left(\tilde{D}_z - s^{-1}\tilde{W}_z\right)$, where $\tilde{W}_z$ is a zero order operator. Then

$$\frac{d\tilde{\mathbb{B}}_s}{ds} e^{-\tilde{\mathbb{B}}^2_s}{}_{|z} = \frac{1}{2}s^{-\frac{1}{2}}\left(\tilde{D}_z - s^{-1}\tilde{W}_z\right)\left[e^{-s\tilde{D}_z^2} + \sum_{q>0}(-s)^q \int_{\triangle_q} A_q d\sigma_1 \dots \sigma_q\right] \tag{17}$$

where $A_q = e^{-\sigma_0 t\tilde{D}_z^2}\tilde{\mathcal{F}}_{[+]}e^{-\sigma_1 t\tilde{D}_z^2}\dots\tilde{\mathcal{F}}_{[+]}e^{-\sigma_q t\tilde{D}_z^2}$.

Consider now the family in (17). Applying the same method as above also the proof of 5.9 follows easily.

$\square$

5.1 Definition of the ρ-form under uniform invertibility hypothesis

Under the hypothesis (**ip1**) of uniform invertibility we can now define

$$\hat{\eta}_{(2)}(\mathcal{D}, r : M \to B\Gamma) := \int_0^\infty \operatorname{Str}_\Gamma\left(\frac{d\tilde{\mathbb{B}}_s}{ds}e^{-\tilde{\mathbb{B}}_s^2}ds\right) \in \mathcal{C}^\infty(B, \Lambda T^*B) \tag{18}$$

and

$$\hat{\rho}(M, \tilde{M}, \mathcal{D}) := \hat{\eta}_{(2)} - \hat{\eta} \in \mathcal{C}^\infty(B, \Lambda T^*B) \tag{19}$$

Proposition 5.10 — *The form $\hat{\rho}(M, \tilde{M}, \mathcal{D})$ is closed.*

Proof— From the transgression formula and the asymptotic behaviour, we get

$$d\hat{\eta} = \int_{\mathrm{M/B}} \hat{A}(M/B)\operatorname{ch}(E/S) = d\hat{\eta}_{(2)}$$

hence $\hat{\rho}$ represents a class in $H^*(B)$. $\square$

We can obtain numbers pairing $\hat{\rho}$ with homology classes $[\omega] \in H_*(B)$.

In section 7 we will extend this definition removing the hypothesis (**ip1**) and requiring some weaker conditions.

6 Fibrations of spin manifolds and PSC questions

Consider a fibration $\pi : M \to B$ s.t. the fibres are Spin manifolds of even dimension, and let $\mathcal{D} = (D_z)_{z\in B}$ be the family of Dirac operators. The invertibility hypothesis (**ip1**) can be satisfied by requiring that the metrics on the fibres are of positive scalar curvature. Now, keeping in mind the results about the Cheeger-Gromov $\rho_{(2)}$ recalled in the introduction in **(a)**, we now ask the following question: *can we use $\hat{\rho}_{(2)}$ to describe the space of metrics on the vertical tangent bundle that are positive scalar curvature?* Let's fix some notation for the study of the geometry of a fibration. $T(M/B) \subset TM$ is the vertical tangent bundle, which will be also denoted with $\mathcal{V}$. Let $g_{M/B}$ be a metric on the vertical bundle. $\forall z \in B$ let $i_z : M_z \to M$ be the natural immersion of the fibre: then $i_z^* g_{M/B}$ is a metric on TM_z. $g_{M/B}$ is in fact a family of metrics on the fibres (we will also use the notation $g_{M/B} = \hat{g} = (\hat{g}_b)_{b\in B}$). When a horizontal distribution $\mathcal{H}$ is fixed, in this section we will denote with $\check{g}$ a horizontal metric.

We can consider the following space:

$$\mathcal{R}^+(M/B) := \{\hat{g} \text{ metric on } T(M/B) \text{ s.t. } \operatorname{scal}\hat{g}_b > 0 \;\; \forall b \in B\} \tag{20}$$

Observe that in the situation of the family $\pi : M \to B$ and the family of coverings $q : \tilde{M} \to B$ with a metric $\hat{g}$ on the vertical tangent is fixed, by Lichnerowicz formula $\forall z \in B$ $D_z^2 = \nabla^*\nabla + \frac{1}{4}\operatorname{scal}(\hat{g}_z)$: hence to guarantee the hypothesis (**ip1**) it is enaugh to require for example $\operatorname{scal}(i_z^* g_{M/B}) \geq \delta > 0$. (If B is compact this is easy to obtain, provided the fibre has PSC).

Let's define the three natural relations in $\mathcal{R}^+(M/B)$, following [PS2]. We say that $\hat{g}_0$ and $\hat{g}_1$ are *path-connected* in $\mathcal{R}^+(M/B)$ if there exists a continuos path $\hat{g}_t \in \mathcal{R}^+(M/B)$ connecting them. Then $\forall t \in [0,1]$ $g^t_{M/B} \in \mathcal{R}^+(M/B)$, and $\forall z \in B$ $i_z^* g^t_{M/B}$ is a path in $\mathcal{R}^+(M_z)$. In ths setting it is natural to consider the family of cylinders $(M_z \times I)_{z\in B}$ (family of manifolds with boundary), fibres of $M \times I \to B$. One has $T((M\times I)/B)_{(m,i_0)} = T(M/B)_m \times T_{i_0}$.

We say that $\hat{g}_0$ and $\hat{g}_1$ are *concordant* if on the fibration of the cylinders $M \times I \to B$ there exists a vertical metric $\hat{G}$ s.t on each fibre $\hat{G}_b$ is of product-type near the boundary, $\mathrm{scal}(\hat{G}_b) > 0$, and on $M \times \{i\} \to B$ it coincides with $\hat{g}_i$, $i = 0,1$. We say that $\hat{g}_0$ and $\hat{g}_1$ are *bordant* if there exists a fibration $W \to B$ of manifolds with boundary with a metric $\hat{G}$ of product-type near the boundaries and of positive scalar curvature along the fibres s.t $\partial W_b = (M_b, \hat{g}_{0,b}) \cup (M_b, \hat{g}_{1,b})$.

Recall that if $\mathcal{H}$ is a fixed horizontal distribution, and $\check{g}$ is a metric on B and $\hat{g}$ is a metric on $\mathcal{V}$, then $g = \hat{g} \oplus \check{g}$ makes M into a Riemannian submersion. Let $t > 0$ and consider $g_t := t\hat{g} \oplus \check{g}$: such rescaled metric is called the *canonical variation* of g. The map $\pi : (M, g_t) \to (B, \check{g})$ is still a Riemannian submersion. We are now interested in the following question: are there relations between the spaces $\mathcal{R}^+(M/B)$ and $\mathcal{R}^+(M)$? Let's recall the relation between $\mathrm{scal}\, g$ and $\mathrm{scal}\, \hat{g}_b$, $\mathrm{scal}\, \check{g}$ (see for example [Be], chapter 9). Let's denote with $\mathcal{H}$ and $\mathcal{V}$ the projections on the horizontal and vertical distributions. Introduce the tensors T, A given by

$$T_{E_1} E_2 := \mathcal{H}(D_{\mathcal{V}E_1} \mathcal{V}E_2) + \mathcal{V}(D_{\mathcal{V}E_1} \mathcal{H}E_2) \tag{21}$$

$$A_{E_1} E_2 := \mathcal{H}(D_{\mathcal{H}E_1} \mathcal{V}E_2) + \mathcal{V}(D_{\mathcal{H}E_1} \mathcal{H}E_2) \tag{22}$$

Fix $\{U_j\}$ a orthonormal base of $\mathcal{V}_x$ and $\{X_i\}$ a orthonormal base for $\mathcal{H}_x$. Define the horizontal vector field $N = \sum_j T_{U_j} U_j$ (which is the mean curvature along each fibre). Moreover for any tensor field E on M define $\check{\delta} E = -\sum_i (D_{X_i} E)_{X_i}$.

Scalar curvature for the canonical variation g_t From [Be], Cor. 9.37 we have that if $s = \mathrm{scal}(g)$, $\hat{s} = \mathrm{scal}\, \hat{g}$, $\check{s} = \mathrm{scal}\, \check{g}$, then

$$s = \hat{s} + \check{s} \circ \pi - |A|^2 - |T|^2 - |N|^2 - 2\check{\delta} N \tag{23}$$

where $|A|^2 := \sum_{ij} g(A_{X_i} U_j, A_{X_i} U_j)$, $|T|^2 := \sum_{ij} g(T_{U_j} X_i, T_{U_j} X_i)$. Then from easy computations (applying lemma 6.9 in [Be]) we get

$$\mathrm{scal}(g_t) = \frac{1}{t}\hat{s} + \check{s} \circ \pi - t|A|^2 - |T|^2 - |N|^2 - 2\check{\delta} N \tag{24}$$

Then it is clear that the following lemma holds

LEMMA 6.11 — *Let $\pi : M \to B$ be a smooth fibration with compact base B. Let $\hat{g}$ be a metric on $\mathcal{V}$, $\hat{g} \in \mathcal{R}^+(M/B)$. Then there exists on M a metric of positive scalar curvature.*

PROOF — Fix a horizontal distribution $\mathcal{H}$ and consider a metric $\check{g}$ on it. Let $g = \hat{g} \oplus \check{g}$ and take its canonical variation g_t. Since B is compact, for t small enaugh in (24) we get $\mathrm{scal}(g_t) > 0$. □

6.1 $\hat{\rho}$ is constant on the connected components of $\mathcal{R}^+(M/B)$

PROPOSITION 6.12 — *Let $\pi : M \to B$ be a fibration with Spin fibres of even dimension.*

Let B be compact without boundary. Let Γ be a discrete group and $\tilde{M} \to B$ the fibration of Γ-coverings. Then the form $\hat{\rho}(M, \tilde{M}, \mathcal{D})$ defined in (19) *is constant on the connected components of $\mathcal{R}^+(M/B)$.*

PROOF— Let $\hat{g}_\lambda \subset \mathcal{R}^+(M/B)$, $\lambda \in [0,1]$ a path connecting $\hat{g}_0$ and $\hat{g}_1$. $\forall z \in B$ let h_z be the metric on the cylinder $M_z \times I$ given by $(\hat{g}_\lambda)_z$: h_z is of product-type near the boundary. Consider the family of Dirac operators on the cylinders $(M_z \times I)_{z \in B}$. The boundary family is formed of the two families $\mathcal{D}^0 = (D_z, \hat{g}_{0,z})_{z\in B}$ and $\mathcal{D}^1 = (D_z, \hat{g}_{1,z})_{z\in B}$, and both of them are families of uniformly invertible operators. Then the Bismut-Cheeger theorem can be applied to have

$$\mathrm{Ch}(\mathcal{I}nd\,\mathcal{D}_{M\times I,h}) = \int_{fibre} AS - \frac{1}{2}\hat{\eta}(M \to B, \hat{g}_0) + \frac{1}{2}\hat{\eta}(M \to B, \hat{g}_1) \quad \text{in} \quad H^*(B, \mathbb{R})$$

Now we shrink the metric on the cylinders to obtain invertible operators: take $h'_z = tg_\lambda \oplus ds^2$ so that $\mathrm{scal}(h'_z) > 0 \ \ \forall z \in B$. By construction h'_z coincides with $t\hat{g}_0$ and $t\hat{g}_1$ on the boundaries. Now $\mathrm{Ch}(\mathcal{I}nd\,\mathcal{D}_{M\times I,h'}) = 0$ so that

$$0 = \int_{fibre} AS - \frac{1}{2}\hat{\eta}(M \to B, t\hat{g}_0) + \frac{1}{2}\hat{\eta}(M \to B, t\hat{g}_1) \quad \text{in} \quad H^*(B, \mathbb{R})$$

Now consider the family of coverings $\tilde{M}_z$. We can reason as before pulling back on the covering the path of metrics. What we get applying the index theorem in [LP] is

$$0 = \int_{fibre} AS - \frac{1}{2}\hat{\eta}_{(2)}(M \to B, \tilde{M}, t\hat{g}_0) + \frac{1}{2}\hat{\eta}_{(2)}(M \to B, \tilde{M}, t\hat{g}_1) \quad \text{in} \quad H^*(B, \mathbb{R})$$

□

Here it would be interesting to use $\hat{\rho}$ or the numbers obtained by pairings with homology classes of the base B to describe the space $\mathcal{R}^+(M/B)$, following ideas of [PS2], [BG].

7 The ρ-form under a weaker hypothesis on the spectrum

In this section we remove the assumption (**ip1**) of uniform invertibility for the operators $(\tilde{D}_b)_{b\in B}$ and we construct the ρ-form for a wider class of problems. It is clear that the delicate point is the $t \to \infty$-asymptotic. We follow [HL], where this asymptotic is developed in the case of a foliation of a compact manifold. In our case $\tilde{M}$ is not compact, but it is bounded geometry, so that computations in [HL] can be easily applied to our different but less complicated situation: we extend them to the definition of $\hat{\rho}$.

It is known that the $t \to \infty$-asymptotic of the heat kernel is deeply connected with the behaviour of the spectrum near zero. (see Appendix E).

Let $\tilde{\mathcal{D}}$ be the family of Dirac operators on the covering and let $\tilde{\mathbb{B}}$ the superconnection adapted. In what follows , we work fibrewise so that all conditions are meant to be fibre by fibre. Let $\tilde{P} = (P_b)_{\beta \in B}$ the projection onto the kernel $\ker \tilde{\mathcal{D}} = \ker \tilde{\mathcal{D}}^2$. Let $\tilde{P}_\epsilon$ be the family of spectral projections of $\tilde{\mathcal{D}}$ relative to the interval $(0, \epsilon)$. We have that $\tilde{P}_\epsilon = \tilde{E}_\epsilon - \tilde{E}_0$, if $\tilde{E}_\lambda$ is the spectral measure associated to the operator $\tilde{\mathcal{D}}^2$ and $\tilde{Q}_\epsilon = 1 - \tilde{P}_\epsilon - \tilde{P}$ relative to $[\epsilon, \infty)$. We start requiring the following regularity hypothesis:

1. $\tilde{P}$, $\tilde{P}_\epsilon$ are smooth w.r.t. $z \in B$
2. $\mathrm{tr}_\Gamma(\tilde{P}_\epsilon) = \mathcal{O}(\epsilon^\beta)$ with $\beta > 3(\dim B + 1)$

Observe that $\mathrm{tr}_\Gamma(\tilde{P}_\epsilon) = \mathrm{tr}_\Gamma(\tilde{E}_\epsilon - \tilde{E}_0) = \mathrm{tr}_\Gamma(\tilde{E}_\epsilon) - b^{(2)}$, where $b^{(2)} = \dim_\Gamma(\mathrm{Ker}\,\tilde{D})$, so that in the case when $\tilde{\mathcal{D}}$ is the family of signature operators, we are requiring that the Novikov-Shubin invariant is greater than $3(\dim B + 1)$.

A simpler case is when $\tilde{P}_\epsilon = 0$ for some $\epsilon > 0$. In this case 0 is uniformly isolated in the spectra of $(\tilde{D}_b)_{b\in B}$ (and correspondingly $\beta = \infty$). We will discuss it.

Recall that $\mathcal{M} = C^\infty(B, \Lambda T^*B \otimes \mathrm{End}\,\tilde{\mathcal{E}})$ is filtered by $\mathcal{M}_i$ = sections of $\sum_{j\geq i} \Lambda^j T^*B \otimes \mathrm{End}\,\tilde{\mathcal{E}})$. In the same way the space $\mathcal{N}$ of sections of the infinite dimensional bundle $\mathcal{K}(\tilde{\mathcal{E}}) \otimes \Lambda T^*B$ is filtered by $\mathcal{N}_i$ = sections of $\sum_{j\geq i} \Lambda^j T^*B \otimes \mathcal{K}(\tilde{\mathcal{E}})$. Since in the most part in this subsection we are dealing with the family of operators on the covering, to simplify the notations let's call only for this section $\tilde{\mathcal{D}} = D$ and remove all tildes.

Let $\mathbb{B}$ the Bismut superconnection and let $\nabla = P\mathbb{B}_{[1]}P$. Then $e^{-t\nabla^2}$ is well defined $\in \mathcal{N}$. Pose

$$\mathbb{B}_\epsilon := (P + Q_\epsilon)\mathbb{B}(P + Q_\epsilon) + P_\epsilon \mathbb{B} P_\epsilon$$

$$A_{\epsilon,t} = \mathbb{B} - \mathbb{B}_\epsilon$$

The rescaled operators are

$$\mathbb{B}_{\epsilon,t} = (P + Q_\epsilon)(\mathbb{B}_t - \sqrt{t}D)(P + Q_\epsilon) + \sqrt{t}D + P_\epsilon(\mathbb{B}_t - \sqrt{t}D)P_\epsilon \tag{25}$$

$$A_{\epsilon,t} = (P + Q_\epsilon)(\mathbb{B}_t - \sqrt{t}D)P_\epsilon + P_\epsilon(\mathbb{B}_t - \sqrt{t}D)(P + Q_\epsilon)$$

Denote also $T_\epsilon = Q_\epsilon \mathbb{B} Q_\epsilon$ and $T_{\epsilon,t} = Q_\epsilon \mathbb{B}_t Q_\epsilon$. Let's recall the diagonalization lemma for the term $\mathbb{B}_\epsilon$ (proved in [HL]).

LEMMA 7.13 — *([HL], prop. 6) $\exists g_\epsilon \in \mathcal{M}$, with $g_\epsilon \in 1 + \mathcal{N}_1$ s.t.*

$$g_\epsilon \mathbb{B}_\epsilon g_\epsilon^{-1} = \begin{vmatrix} \nabla^2 & 0 & 0 \\ 0 & T_\epsilon^2 & 0 \\ 0 & 0 & P_\epsilon \mathbb{B} P_\epsilon \end{vmatrix} \quad mod \quad \begin{vmatrix} \mathcal{N}_3 & 0 & 0 \\ 0 & \mathcal{N}_2 & 0 \\ 0 & 0 & 0 \end{vmatrix}.$$

From this lemma we get $\mathbb{B}_{\epsilon,t}^2 = t\delta_t \mathbb{B}_\epsilon^2 \delta_t^{-1} =$

$$= t\delta_t g_\epsilon^{-1} \left(\begin{vmatrix} \nabla^2 & 0 & 0 \\ 0 & T_\epsilon^2 & 0 \\ 0 & 0 & P_\epsilon \mathbb{B}_t P_\epsilon \end{vmatrix} + \begin{vmatrix} \mathcal{N}_3 & 0 & 0 \\ 0 & \mathcal{N}_2 & 0 \\ 0 & 0 & 0 \end{vmatrix} \right) g_\epsilon \delta_t =$$

$$= \delta_t g_\epsilon^{-1} \delta_t^{-1} \begin{pmatrix} t\delta_t(\nabla^2 + \mathcal{N}_3)\delta_t^{-1} & 0 & 0 \\ 0 & t\delta_t(T_\epsilon^2 + \mathcal{N}_2)\delta_t^{-1} & 0 \\ 0 & 0 & P_\epsilon \mathbb{B}_t P_\epsilon \end{pmatrix} \delta_t g_\epsilon \delta_t^{-1}$$

Now observe that $\nabla^2 \in \mathcal{M}_2$ so that $t\delta_t \nabla^2 \delta_t^{-1} = \nabla^2$. To estimate the residual terms belonging to $\mathcal{N}_k$ we recall the following lemma, proved in [HL].

LEMMA 7.14 — *If $A \in \mathcal{N}_k$ is a residual term in the diagonalization lemma or is a term in $g_\epsilon - 1$ or g_ϵ^{-1}, then, posing $\epsilon = t^{-\frac{1}{a}}$, $A_t := \delta_t A \delta_t^{-1}$ verifies: $\forall r, s$*

$$\|A_t\|_{r,s} = \mathcal{O}(t^{-\frac{k}{2}+\frac{k}{a}}) \quad as \quad t \to \infty$$

Then at place (1,1) in the diagonalized matrix above we get $\nabla^2 + \mathcal{O}(t^{-\frac{3}{2}+\frac{3}{a}+1}) = \mathcal{O}(t^{-\frac{1}{2}+\frac{3}{a}})$. To have $-\frac{1}{2}+\frac{3}{a} < 0$ we take $a > 6$. The term at place (2,2) gives $T^2_{\epsilon,t} + \mathcal{O}(t^{\frac{2}{a}})$. Then

$$\mathbb{B}^2_{\epsilon,t} = \delta_t g_\epsilon^{-1}\delta_t^{-1} \begin{vmatrix} \nabla^2 + \mathcal{O}(t^{-\alpha}) & 0 & 0 \\ 0 & T^2_\epsilon + \mathcal{O}(t^{\frac{2}{a}}) & 0 \\ 0 & 0 & P_\epsilon \mathbb{B} P_\epsilon \end{vmatrix} \delta_t g_\epsilon \delta_t^{-1} \quad \text{with} \quad \alpha > 0$$

Now since from diagonalization lemma $g_\epsilon = g_{n+1} \oplus 1$

$$\mathbb{B}^2_{\epsilon,t} = \left(\begin{array}{c|c} \delta_t g_{n+1}^{-1}\delta_t^{-1} \begin{vmatrix} \nabla^2 + \mathcal{O}(t^{-\alpha}) & 0 \\ 0 & T^2_{\epsilon,t} + \mathcal{O}(t^{\frac{2}{a}}) \end{vmatrix} \delta_t g_{n+1}\delta_t^{-1} & 0 \\ \hline 0 & P_\epsilon \mathbb{B} P_\epsilon \end{array} \right)$$

Observe that since $g_\epsilon - 1, g_\epsilon^{-1} - 1 \in \mathcal{N}_1$,

$$\delta_t g_{n+1}^{-1}\delta_t^{-1} = 1 + \begin{vmatrix} \mathcal{O}(t^{-\frac{1}{2}+\frac{1}{a}}) & \mathcal{O}(t^{-\frac{1}{2}+\frac{1}{a}}) \\ \mathcal{O}(t^{-\frac{1}{2}+\frac{1}{a}}) & \mathcal{O}(t^{-\frac{1}{2}+\frac{1}{a}}) \end{vmatrix}$$

The case of 0 uniformly isolated. Suppose that there exists $\epsilon_0 > 0$ s.t. $P_{\epsilon_0} = 0$. In this case $\mathbb{B}_t = \mathbb{B}_{t,\epsilon}$. Pose as before $\epsilon = t^{-\frac{1}{a}}$, with $a > 6$. Denote $\theta := \mathcal{O}(t^{-\frac{1}{2}+\frac{1}{a}})$ and $e^{-T} = e^{-(T^2_{\epsilon,t}+\mathcal{O}(t^{\frac{2}{a}}))}$ for $e^{-\mathbb{B}^2_t}$ we get the expression

$$e^{-\mathbb{B}^2_t} = \begin{vmatrix} 1+\theta & \theta \\ \theta & 1+\theta \end{vmatrix} \begin{vmatrix} e^{-\nabla^2} + \mathcal{O}(t^{-\alpha}) & 0 \\ 0 & e^{-T} \end{vmatrix} \begin{vmatrix} 1+\theta & \theta \\ \theta & 1+\theta \end{vmatrix} =$$

$$= \underbrace{\begin{vmatrix} 1+\theta & \theta \\ \theta & 1+\theta \end{vmatrix} \begin{vmatrix} e^{-\nabla^2} & 0 \\ 0 & 0 \end{vmatrix} \begin{vmatrix} 1+\theta & \theta \\ \theta & 1+\theta \end{vmatrix}}_{A} +$$

$$+ \underbrace{\begin{vmatrix} 1+\theta & \theta \\ \theta & 1+\theta \end{vmatrix} \begin{vmatrix} \mathcal{O}(t^{-\alpha}) & 0 \\ 0 & e^{-T} \end{vmatrix} \begin{vmatrix} 1+\theta & \theta \\ \theta & 1+\theta \end{vmatrix}}_{B} =$$

$$= A + B.$$

$$A = \begin{vmatrix} (1+\theta)^2 e^{-\nabla^2} & \theta(1+\theta)e^{-\nabla^2} \\ \theta(1+\theta)e^{-\nabla^2} & \theta^2 e^{-\nabla^2} \end{vmatrix} = \begin{vmatrix} e^{-\nabla^2} & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} \mathcal{O}(t^{-1+\frac{2}{a}}) & \mathcal{O}(t^{-\frac{1}{2}+\frac{1}{a}}) \\ \mathcal{O}(t^{-\frac{1}{2}+\frac{1}{a}}) & \mathcal{O}(t^{-1+\frac{2}{a}}) \end{vmatrix}$$

$$B = \begin{vmatrix} (1+\theta)^2 \mathcal{O}(t^{-\alpha}) & \theta(1+\theta)[\mathcal{O}(t^{-\alpha}) + e^{-T}] \\ \theta(1+\theta)[\mathcal{O}(t^{-\alpha}) + e^{-T}] & \theta^2\mathcal{O}(t^{-\alpha}) + (1+\theta)^2 e^{-T} \end{vmatrix}$$

so that

$$A + B = \begin{vmatrix} e^{-\nabla^2} & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} \mathcal{O}(t^{-1+\frac{2}{a}}) & \mathcal{O}(t^{-\frac{1}{2}+\frac{1}{a}}) \\ \mathcal{O}(t^{-\frac{1}{2}+\frac{1}{a}}) & \mathcal{O}(t^{-1+\frac{2}{a}}) \end{vmatrix}$$

Clearly

$$\lim_{t\to\infty} \mathrm{Str}_\Gamma(e^{-\tilde{\mathbb{B}}^2_t}) = \mathrm{Str}_\Gamma e^{-\nabla^2} = \mathrm{Ch}\ \mathcal{I}\mathrm{nd}_\Gamma(\mathcal{D})$$

It is also easy to give an estimate for $\mathrm{Str}_\Gamma(\frac{d\mathbb{B}_t}{dt}e^{-\mathbb{B}^2_t})$. In fact

$$\frac{d\mathbb{B}_t}{dt}e^{-\mathbb{B}^2_t} = \left(\begin{vmatrix} 0 & 0 \\ 0 & t^{-\frac{1}{2}}D \end{vmatrix} + \mathcal{O}(t^{-\frac{3}{2}}) \right)(A+B) =$$

$$= \begin{vmatrix} 0 & 0 \\ \mathcal{O}(t^{-1+\frac{2}{a}}) & \mathcal{O}(t^{-\frac{3}{2}+\frac{2}{a}}) \end{vmatrix} + \begin{vmatrix} \mathcal{O}(t^{-\frac{3}{2}}) & 0 \\ \mathcal{O}(t^{-\frac{3}{2}}) & 0 \end{vmatrix} + \mathcal{O}(t^{-2}) = \begin{vmatrix} \mathcal{O}(t^{-\frac{3}{2}}) & \mathcal{O}(t^{-2}) \\ \mathcal{O}(t^{-1+\frac{2}{a}}) & \mathcal{O}(t^{-\frac{3}{2}+\frac{2}{a}}) \end{vmatrix}$$

Since $\frac{2}{a} \leq \frac{1}{3} < \frac{1}{2}$ and since only diagonal blocks contribute[2] to the Str_Γ, it is clear that in the case $P_\epsilon = 0$ the form $\beta(t) = \mathrm{Str}_\Gamma\left(\frac{d\mathbb{B}_t}{dt}e^{-\mathbb{B}_t^2}\right)$ is integrable for $t \to \infty$.

The general case ($P_\epsilon \neq 0$). Recall we have posed in (25) $\mathbb{B}_t = \mathbb{B}_{\epsilon,t} + A_{\epsilon,t}$. Now to express $e^{-\mathbb{B}_t^2}$ write $\mathbb{B}_t(z) = \mathbb{B}_{t,\epsilon} + zA_{t,\epsilon}$ so that

$$e^{-\mathbb{B}_t^2} - e^{-\mathbb{B}_{t,\epsilon}^2} = \int_0^1 \frac{d}{dz} e^{-\mathbb{B}_t(z)^2} dz =$$

equals, from proposition 3.10 in [HL],

$$-\int_0^1 \int_0^1 e^{-(s-1)\mathbb{B}_t^2(z)} \frac{d\mathbb{B}_t^2(z)}{dz} e^{-s\mathbb{B}_t^2(z)} ds dz =: F_{\epsilon,t}$$

It is proved in [HL] that $\lim_{t\to\infty} \mathrm{Str}_\Gamma\left(e^{-\tilde{\mathbb{B}}_t^2}\right) = \mathrm{Str}_\Gamma e^{-\nabla^2}$. Now we examine the integrability for $t \to \infty$ of

$$\mathrm{Str}_\Gamma(\frac{d\mathbb{B}_t}{dt}e^{-\mathbb{B}_t^2}) = \mathrm{Str}_\Gamma(\frac{d\mathbb{B}_{t,\epsilon}}{dt}e^{-\mathbb{B}_{t,\epsilon}^2}) + \mathrm{Str}_\Gamma(\frac{d\mathbb{B}_{t,\epsilon}}{dt}F_{\epsilon,t}) \tag{26}$$

Let's study separately the two terms $\mathrm{Str}_\Gamma(\frac{d\mathbb{B}_{t,\epsilon}}{dt}e^{-\mathbb{B}_{t,\epsilon}^2})$ and $\mathrm{Str}_\Gamma(\frac{d\mathbb{B}_{t,\epsilon}}{dt}F_{\epsilon,t})$. For the first term:

$$\frac{d\mathbb{B}_{t,\epsilon}}{dt}e^{-\mathbb{B}_{t,\epsilon}^2} = \left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & t^{-\frac{1}{2}}T_\epsilon & 0 \\ 0 & 0 & t^{-\frac{1}{2}}P_\epsilon D P_\epsilon \end{pmatrix} + \mathcal{O}(t^{-\frac{3}{2}})\right] \begin{pmatrix} e^{-\nabla^2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e^{-P_\epsilon \mathbb{B}_t^2 P_\epsilon} \end{pmatrix} +$$

$$+ \begin{pmatrix} \mathcal{O}(t^{-1+\frac{2}{a}}) & \mathcal{O}(t^{-\frac{1}{2}+\frac{1}{a}}) & 0 \\ \mathcal{O}(t^{-\frac{1}{2}+\frac{1}{a}}) & \mathcal{O}(t^{-1+\frac{2}{a}}) & 0 \\ 0 & 0 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & t^{-\frac{1}{2}}P_\epsilon D P_\epsilon e^{-P_\epsilon \mathbb{B}_t^2 P_\epsilon} \end{pmatrix} + \begin{pmatrix} \mathcal{O}(t^{-\frac{3}{2}})e^{-\nabla^2} & 0 & \mathcal{O}(t^{-\frac{3}{2}})e^{-P_\epsilon \mathbb{B}_t^2 P_\epsilon} \\ \mathcal{O}(t^{-\frac{3}{2}})e^{-\nabla^2} & 0 & \mathcal{O}(t^{-\frac{3}{2}})e^{-P_\epsilon \mathbb{B}_t^2 P_\epsilon} \\ \mathcal{O}(t^{-\frac{3}{2}})e^{-\nabla^2} & 0 & \mathcal{O}(t^{-\frac{3}{2}})e^{-P_\epsilon \mathbb{B}_t^2 P_\epsilon} \end{pmatrix} +$$

$$+ \begin{pmatrix} 0 & 0 & 0 \\ \mathcal{O}(t^{-1+\frac{2}{a}}) & \mathcal{O}(t^{-\frac{3}{2}+\frac{2}{a}}) & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} \mathcal{O}(t^{-2+\frac{2}{a}}) & \mathcal{O}(t^{-2+\frac{2}{a}}) & 0 \\ \mathcal{O}(t^{-2+\frac{2}{a}}) & \mathcal{O}(t^{-2+\frac{2}{a}}) & 0 \\ \mathcal{O}(t^{-2+\frac{2}{a}}) & \mathcal{O}(t^{-2+\frac{2}{a}}) & 0 \end{pmatrix}$$

so that when taking the supertrace, we get that we only have to guarantee the integrability of $\mathrm{Str}_\Gamma(t^{-\frac{1}{2}}P_\epsilon D P_\epsilon e^{-P_\epsilon \mathbb{B}_t^2 P_\epsilon})$ and $\mathrm{Str}_\Gamma(\mathcal{O}(t^{-\frac{3}{2}})e^{-P_\epsilon \mathbb{B}_t^2 P_\epsilon})$. The latter is integrable; for the other part, which correctly written is $\mathrm{Str}_\Gamma(t^{-\frac{1}{2}}P_\epsilon D P_\epsilon e^{-P_\epsilon \mathbb{B}_t^2 P_\epsilon} P_\epsilon)$, we reason as follows

$$\mathrm{Str}_\Gamma(t^{-\frac{1}{2}}P_\epsilon D P_\epsilon e^{-P_\epsilon \mathbb{B}_t^2 P_\epsilon}) = t^{-\frac{1}{2}}\,\mathrm{tr}_\Gamma(UP_\epsilon)$$

where $U = \eta P_\epsilon D P_\epsilon e^{-P_\epsilon \mathbb{B}_t^2 P_\epsilon}$ and η is the grading s.t $\mathrm{Str}\, A = \mathrm{tr}(\eta A)$. Now we evaluate $\mathrm{tr}_\Gamma(UP_\epsilon) = \mathrm{tr}_\Gamma(UP_\epsilon^2) = \mathrm{tr}_\Gamma(P_\epsilon U P_\epsilon)$. Let $\omega_1, \ldots, \omega_J$ a base of $\Lambda T_z^* B$, for z fixed on the base B. U is a family of operators and U_z acts on $\mathbb{C}^\infty(\tilde{M}_z, \tilde{E}_z) \otimes \Lambda T_z^* B$. Write $U_z = \sum_j U_j \otimes \omega_j$. Now

$$\mathrm{tr}_\Gamma(P_\epsilon U P_\epsilon) = \sum_j \mathrm{tr}_\Gamma(P_\epsilon U_j P_\epsilon) \otimes \omega_j =$$

[2]In fact if P_i are projections s.t. $\sum_i P_i = 1$, then for an operator A we have $\mathrm{Str}\, A = \mathrm{tr}\, \eta A = \mathrm{tr}(\sum_i P_i \eta A P_i) + \mathrm{tr}(\sum_{i\neq j} P_i \eta A P_j) = \mathrm{tr}(\sum_i P_i \eta A P_i)$ since tr is zero on commutators.

recalling we are on the coverings

$$= \sum_j \operatorname{tr}(\chi_{\mathcal{F}} P_\epsilon U_j P_\epsilon \chi_{\mathcal{F}}) \otimes \omega_j \tag{27}$$

Now

$$\operatorname{tr}(\chi_{\mathcal{F}} P_\epsilon U_j P_\epsilon \chi_{\mathcal{F}}) = \sum_i < \chi_{\mathcal{F}} P_\epsilon U_j P_\epsilon \chi_{\mathcal{F}} \delta_{v_i}, \delta_{v_i} >=$$

(where $\{\delta_{v_i}\}$ is a base of $L^2(\tilde{M}_{z|\mathcal{F}}, \tilde{E}_{z|\mathcal{F}})$)

$$= \sum_i < U_j P_\epsilon \chi_{\mathcal{F}} \delta_{v_i}, P_\epsilon \chi_{\mathcal{F}} \delta_{v_i} >$$

$$| < U_j P_\epsilon \chi_{\mathcal{F}} \delta_{v_i}, P_\epsilon \chi_{\mathcal{F}} \delta_{v_i} > | \leq \|U_j P_\epsilon \chi_{\mathcal{F}} \delta_{v_i}\| \cdot \|P_\epsilon \chi_{\mathcal{F}} \delta_{v_i}\| \leq$$

$$\leq \|U_j\| \, \|P_\epsilon \chi_{\mathcal{F}} \delta_{v_i}\|^2 \leq \|U_z\| \, \|P_\epsilon \chi_{\mathcal{F}} \delta_{v_i}\|^2$$

$$\sum_i \|P_\epsilon \chi_{\mathcal{F}} \delta_{v_i}\| = \sum_i < P_\epsilon \chi_{\mathcal{F}} \delta_{v_i}, P_\epsilon \chi_{\mathcal{F}} \delta_{v_i} >=$$

$$= \sum_i < \chi_{\mathcal{F}} P_\epsilon \chi_{\mathcal{F}} \delta_{v_i}, \delta_{v_i} >= \operatorname{tr}_\Gamma(P_\epsilon) = \mathcal{O}(\epsilon^\beta)$$

Hence

$$\operatorname{tr}_\Gamma(P_\epsilon U P_\epsilon) \leq \|U\| \, \mathcal{O}(\epsilon^\beta) = \|U\| \, \mathcal{O}(t^{-\frac{\beta}{a}}) \ \ , \ \text{ with } \ \epsilon = t^{-\frac{1}{a}}$$

Observe that the term we want to be integrable in t is $t^{-\frac{1}{2}} \operatorname{tr}_\Gamma(UP_\epsilon) \leq c \, \|U\| \, t^{-\frac{\beta}{a}-\frac{1}{2}}$. Now since $U = \eta P_\epsilon D P_\epsilon e^{-P_\epsilon \mathbb{B}_t^2 P_\epsilon}$ is such that $t^{-\frac{q}{2}} \, \|U\|$ is bounded independentely of t for $t \to \infty$, then

$$\operatorname{Str}_\Gamma(\frac{d\mathbb{B}_t}{dt} e^{-\mathbb{B}_{t,\epsilon}^2}) \leq c t^{\frac{q}{2}-\frac{\beta}{a}-\frac{1}{2}} = c t^{\frac{q-1}{2}-\frac{\beta}{a}}$$

We require

$$\frac{q-1}{2} - \frac{\beta}{a} < -1$$

to have integrability hence we need $a < \frac{2\beta}{q+1}$. a is also required to be $a > 6$ (see the beginning of the paragraph about the case of 0 uniformly isolated) so that the hypothesis

$$\beta > 3(q+1) \tag{28}$$

is a *sufficient condition* to have integrability for the first term in (26). Now let's consider the second term in (26) and study

$$\frac{d\mathbb{B}_t}{dt} F_{\epsilon,t} = \frac{1}{2}(t^{-\frac{1}{2}} D - t^{-\frac{3}{2}} \mathbb{B}_{[2]}) \int_0^1 \int_0^1 e^{-(s-1)\mathbb{B}_t^2(z)} \frac{d\mathbb{B}_t^2(z)}{dz} e^{-s\mathbb{B}_t^2(z)} ds dz$$

We have $\dfrac{d\mathbb{B}_t^2(z)}{dz} = \mathbb{B}_t(z) A_{\epsilon,t} + A_{\epsilon,t} \mathbb{B}_t(z) =$

$$\sqrt{t} D A_{\epsilon,t} + (P+Q_\epsilon)H(P+Q_\epsilon)HP_\epsilon + P_\epsilon H P_\epsilon H(P+Q_\epsilon) + z(P+Q_\epsilon)HP_\epsilon H(P+Q_\epsilon) +$$

$$+ z P_\epsilon H(P+Q_\epsilon)HP_\epsilon + A_{\epsilon,t}\sqrt{t} D + P_\epsilon H(P+Q_\epsilon)H(P+Q_\epsilon) + z P_\epsilon H(P+Q_\epsilon)HP_\epsilon +$$

$$+ (P+Q_\epsilon)HP_\epsilon HP_\epsilon + z(P+Q_\epsilon)HP_\epsilon H(P+Q_\epsilon) =$$

where $H = \mathbb{B}_{[1]} + t^{-\frac{1}{2}}\mathbb{B}_{[2]}$

$$= \sqrt{t}DP_\epsilon(\mathbb{B}_{[1]} + t^{-\frac{1}{2}}\mathbb{B}_{[2]})(P+Q_\epsilon) + \sqrt{t}D(P+Q_\epsilon)(\mathbb{B}_{[1]} + t^{-\frac{1}{2}}\mathbb{B}_{[2]})P_\epsilon + (P+Q_\epsilon)(\mathbb{B}_{[1]}+$$

$$+t^{-\frac{1}{2}}\mathbb{B}_{[2]})(P+Q_\epsilon)(\mathbb{B}_{[1]} + t^{-\frac{1}{2}}\mathbb{B}_{[2]})P_\epsilon + P_\epsilon(\mathbb{B}_{[1]} + t^{-\frac{1}{2}}\mathbb{B}_{[2]})P_\epsilon(\mathbb{B}_{[1]} + t^{-\frac{1}{2}}\mathbb{B}_{[2]})(P+Q_\epsilon)+$$

$$+z(P+Q_\epsilon)(\mathbb{B}_{[1]} + t^{-\frac{1}{2}}\mathbb{B}_{[2]})P_\epsilon(\mathbb{B}_{[1]} + t^{-\frac{1}{2}}\mathbb{B}_{[2]})(P+Q_\epsilon)+$$

$$+zP_\epsilon(\mathbb{B}_{[1]} + t^{-\frac{1}{2}}\mathbb{B}_{[2]})(P+Q_\epsilon)(\mathbb{B}_{[1]} + t^{-\frac{1}{2}}\mathbb{B}_{[2]})P_\epsilon+$$

$$+P_\epsilon(\mathbb{B}_{[1]} + t^{-\frac{1}{2}}\mathbb{B}_{[2]})(P+Q_\epsilon)\sqrt{t}D + (P+Q_\epsilon)(\mathbb{B}_{[1]} + t^{-\frac{1}{2}}\mathbb{B}_{[2]})P_\epsilon\sqrt{t}D+$$

$$+P_\epsilon(\mathbb{B}_{[1]} + t^{-\frac{1}{2}}\mathbb{B}_{[2]})(P+Q_\epsilon)(\mathbb{B}_{[1]} + t^{-\frac{1}{2}}\mathbb{B}_{[2]})(P+Q_\epsilon)+$$

$$+zP_\epsilon(\mathbb{B}_{[1]} + t^{-\frac{1}{2}}\mathbb{B}_{[2]})(P+Q_\epsilon)(\mathbb{B}_{[1]} + t^{-\frac{1}{2}}\mathbb{B}_{[2]})P_\epsilon+$$

$$+(P+Q_\epsilon)(\mathbb{B}_{[1]} + t^{-\frac{1}{2}}\mathbb{B}_{[2]})P_\epsilon(\mathbb{B}_{[1]} + t^{-\frac{1}{2}}\mathbb{B}_{[2]})P_\epsilon+$$

$$+z(P+Q_\epsilon)(\mathbb{B}_{[1]} + t^{-\frac{1}{2}}\mathbb{B}_{[2]})P_\epsilon(\mathbb{B}_{[1]} + t^{-\frac{1}{2}}\mathbb{B}_{[2]})(P+Q_\epsilon)$$

Let's analyze the terms in the sum above, keeping in mind that we are going to compute a supertrace. In the term

$$\sqrt{t}DP_\epsilon(\mathbb{B}_{[1]} + t^{-\frac{1}{2}}\mathbb{B}_{[2]})(P+Q_\epsilon)$$

we can reduce to the addend $\sqrt{t}DP_\epsilon\mathbb{B}_{[1]}(P+Q_\epsilon)$ when considering the supertrace, since $\mathbb{B}_{[2]}$ is a family of vertical operators so that observation in the last footnote applies. Now writing in local coordinates $\sqrt{t}DP_\epsilon\mathbb{B}_{[1]}(P+Q_\epsilon) = \sqrt{t}DP_\epsilon\mathbb{B}_{[1]}(1-P_\epsilon) =$

$$= \sqrt{t}DP_\epsilon(d+\Phi)(1-P_\epsilon)$$

where Φ is a family of vertical zero order operators and again by the same argument we can reduce to

$$\sqrt{t}DP_\epsilon d(P_\epsilon) \tag{29}$$

Now examine another element, for example

$$(P+Q_\epsilon)(\mathbb{B}_{[1]} + t^{-\frac{1}{2}}\mathbb{B}_{[2]})(P+Q_\epsilon)(\mathbb{B}_{[1]} + t^{-\frac{1}{2}}\mathbb{B}_{[2]})P_\epsilon \tag{30}$$

Here under supertrace we can reduce to

$$(P+Q_\epsilon)\mathbb{B}_{[1]}(P+Q_\epsilon)\mathbb{B}_{[1]}P_\epsilon \;+\; (P+Q_\epsilon)t^{-\frac{1}{2}}\mathbb{B}_{[2]}(P+Q_\epsilon)\mathbb{B}_{[1]}P_\epsilon+$$

$$+(P+Q_\epsilon)\mathbb{B}_{[1]}(P+Q_\epsilon)t^{-\frac{1}{2}}\mathbb{B}_{[2]}P_\epsilon =$$

$$(1-P_\epsilon)(d+\Phi)(1-P_\epsilon)(d+\Phi)P_\epsilon \;+\; (P+Q_\epsilon)(1-P_\epsilon)t^{-\frac{1}{2}}\mathbb{B}_{[2]}(1-P_\epsilon)\mathbb{B}_{[1]}(d+\Phi)P_\epsilon+$$

$$+(1-P_\epsilon)(d+\Phi)(1-P_\epsilon)t^{-\frac{1}{2}}\mathbb{B}_{[2]}P_\epsilon$$

With this kind of reductions, we get that, since we are going to take a supertrace, we can reduce to consider

$$\frac{d\mathbb{B}_t^2(z)}{dz} = \sqrt{t}DA_1 + A_2\sqrt{t}D + A_3$$

where $A_i = C_{i,1}P_\epsilon C_{i,2}$, with $C_{i,j} \in \mathcal{M}_1$ are sums of words in Φ, $d(\Phi)$, $t^{-\frac{1}{2}}\mathbb{B}_{[2]}$, $t^{-\frac{1}{2}}d(\mathbb{B}_{[2]})$. This implies that $C_{i,j}$ are differential operators with coefficients uniformly bounded in t.

Then $\mathrm{Str}_\Gamma\left(\frac{d\mathbb{B}_t}{dt}F_{\epsilon,t}\right) =$

$$= \mathrm{Str}_\Gamma \frac{1}{2}(t^{-\frac{1}{2}}D - t^{-\frac{3}{2}}\mathbb{B}_{[2]}) \int_0^1 \int_0^1 e^{-(s-1)\mathbb{B}_t^2(z)}(\sqrt{t}DC_{1,1}P_\epsilon C_{1,2}+$$

$$+C_{2,1}P_\epsilon C_{2,2}\sqrt{t}D + C_{3,1}P_\epsilon C_{3,2})e^{-s\mathbb{B}_t^2(z)}dsdz =$$

$$= \mathrm{tr}_\Gamma \Big(\eta\frac{1}{2}(t^{-\frac{1}{2}}D - t^{-\frac{3}{2}}\mathbb{B}_{[2]}) \int_0^1 \int_0^1 e^{-(s-1)\mathbb{B}_t^2(z)}(\sqrt{t}D(C_{1,1}P_\epsilon)P_\epsilon(P_\epsilon C_{1,2}) +$$

$$+ (C_{2,1}P_\epsilon)P_\epsilon(P_\epsilon C_{2,2})\sqrt{t}D + (C_{3,1}P_\epsilon)P_\epsilon(P_\epsilon C_{3,2}))e^{-s\mathbb{B}_t^2(z)}dsdz\Big) =$$

$$= \mathrm{tr}_\Gamma \frac{1}{2}(t^{-\frac{1}{2}}D - t^{-\frac{3}{2}}\mathbb{B}_{[2]}) \int_0^1 \int_0^1 \Big[P_\epsilon C_{1,2}e^{-s\mathbb{B}_t^2(z)}\eta e^{-(s-1)\mathbb{B}_t^2(z)}\sqrt{t}DC_{1,1}P_\epsilon+$$

$$+P_\epsilon C_{2,2}\sqrt{t}De^{-s\mathbb{B}_t^2(z)}\eta e^{-(s-1)\mathbb{B}_t^2(z)}C_{2,1}P_\epsilon + P_\epsilon C_{3,2}e^{-s\mathbb{B}_t^2(z)}\eta e^{-(s-1)\mathbb{B}_t^2(z)}C_{2,1}P_\epsilon\Big] =$$

$= \mathrm{tr}_\Gamma(P_\epsilon W P_\epsilon)$ with

$$W = (t^{-\frac{1}{2}}D - t^{-\frac{3}{2}}\mathbb{B}_{[2]}) \Big[C_{1,2}e^{-s\mathbb{B}_t^2(z)}\eta e^{-(s-1)\mathbb{B}_t^2(z)}\sqrt{t}DC_{1,1}+$$

$$+C_{2,2}\sqrt{t}De^{-s\mathbb{B}_t^2(z)}\eta e^{-(s-1)\mathbb{B}_t^2(z)}C_{2,1} + C_{3,2}e^{-s\mathbb{B}_t^2(z)}\eta e^{-(s-1)\mathbb{B}_t^2(z)}C_{2,1}\Big]$$

Now again as in (27) we have that $t^{-\frac{q}{2}}\left\| e^{-s\mathbb{B}_t^2(z)}\eta e^{-(s-1)\mathbb{B}_t^2(z)}\right\|$ is bounded independently of t for $t \to \infty$ so that the condition (28) on the Novikov-Shubin exponent guaranties that also this second term is integrable.

We have therefore proven the following proposition

PROPOSITION 7.15 — *Let $\tilde{\mathcal{D}}$ be the family of Dirac operators on the covering. Let $\tilde{P} = (P_b)_{b\in B}$ the family of projections onto the kernel $\ker\tilde{\mathcal{D}} = \tilde{\mathcal{D}}^2$. Let $\tilde{P}_\epsilon$ be the family of spectral projections of $\tilde{\mathcal{D}}$ relative to the interval $(0,\epsilon)$ and $\tilde{Q}_\epsilon = 1 - \tilde{P}_\epsilon - \tilde{P}$ relative to $[\epsilon,\infty)$. If the following regularity hypothesis are satisfied*

1. *$\tilde{P}$, $\tilde{P}_\epsilon$ are smooth in z*
2. *$\mathrm{tr}_\Gamma(\tilde{P}_\epsilon) = \mathcal{O}(\epsilon^\beta)$ with $\beta > 3(\dim B + 1)$*

then the form

$$\hat{\rho}(M,\tilde{M},\mathcal{D}) := \hat{\eta} - \hat{\eta}_{(2)} \in \mathcal{C}^\infty(B,\Lambda T^*B)$$

is well defined and it is closed. It gives a class $[\hat{\rho}] \in H^*(B)$.

The local part in subsection 4 does not require regularity hypothesis, so that that $\hat{\rho}$ gives a class in $H^*(B)$ also in this case.

8 Rho-form in the case of odd dimensional fibre

Let $\pi : M \to B$ be a fibration with odd dimensional fibre F. Let $E \to M$ be a vector bundle of vertical Clifford modules as in the setting, which we require to be endowed with a Clifford connection ∇^E. Let $\mathcal{D}$ be the family of Dirac operators on the fibres. Since the fibre is odd dimensional, E is not naturally $\mathbb{Z}_2$-graded. There is anyway a kind of superconnection also in this case (see [Q] or appendix A.3). The basic idea is that a vector bundle is equivlent to a super-bundle with an action of $\mathbb{C}l(1)$.

As in [Q] and [BC1], replace the bundle E by $E\oplus E$, and $\mathcal{D}$ by the diagonal operator $\mathcal{D}^\Delta = \begin{pmatrix} \mathcal{D} & 0 \\ 0 & \mathcal{D} \end{pmatrix}$. We consider $E\oplus E$ as a module over $\mathbb{C}l(1)$ whose generator is $\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$).

Definition 8.16 — A $\mathbb{C}l(1)$-superconnection adapted to $\mathcal{D}$ is a superconnection adapted to $\mathcal{D}^\Delta$ which also commutes with σ (by definition $\mathbb{B}$ is still odd with respect to the total grading of $\mathcal{A}(M, E\oplus E)$)

$$\mathbb{B} : \mathcal{A}(M, E\oplus E) \to \mathcal{A}(M, E\oplus E) \tag{31}$$

which commutes with σ.

The rescaled $\mathbb{C}l(1)$-*Bismut superconnection* is given ([BC1]) by

$$\mathbb{B}_t = \sqrt{t}\mathcal{D}^\Delta\sigma + \tilde{\nabla}^\Delta - \frac{c(T)^\Delta}{4\sqrt{t}}\sigma$$

where $\tilde{\nabla}^\Delta$ is the infinite dimensional unitary connection defined on the superbundle $E\oplus E$.
The Bismut-Cheeger eta form for the family $\mathcal{D}$ is

$$\hat{\eta}(\mathcal{D}) = \frac{1}{\sqrt{\pi}}\int_0^\infty \operatorname{Tr}_{\mathbb{C}l(1)}\left(\frac{d\mathbb{B}_t}{dt}e^{-\mathbb{B}_t^2}\right)dt \tag{32}$$

($\hat{\eta}$ is in this case an even degree form, since $\mathbb{B}_t^2$ and $e^{-\mathbb{B}_t^2}$ are even degree elements of $C^\infty(M, \pi^*\Lambda T^*B\otimes \operatorname{End}(E\oplus E))$), while

$$\frac{d\mathbb{B}_t}{dt} = \frac{1}{2}\left(\frac{1}{\sqrt{t}}\mathcal{D}^\Delta\sigma - \frac{c(T)^\Delta}{4\sqrt{t}^3}\sigma\right) = W^\Delta\sigma$$

is odd and $\operatorname{Tr}_{\mathbb{C}l(1)}$ is an odd operator, as explained in appendix). Now, removing somewhere the $^\Delta$ from diagonal operator to simplify notations, compute

$$\mathbb{B}_t^2 = \tilde{\nabla}^2 + t\mathcal{D} - \left(\frac{c(T)}{4}\right)^2\frac{1}{t} + \sqrt{t}(\tilde{\nabla}\mathcal{D} + \mathcal{D}\tilde{\nabla})\sigma - \frac{1}{4\sqrt{t}}\left(\tilde{\nabla}c(T) + c(T)\tilde{\nabla}\right)\sigma +$$

$$+\frac{1}{4}\left(\mathcal{D}c(T) + c(T)\mathcal{D}\right) = A^\Delta + B^\Delta\sigma$$

By standard computations (see [Q]) give that

$$\operatorname{Tr}_{\mathbb{C}l(1)}\left(\frac{d\mathbb{B}_t}{dt}e^{-\mathbb{B}_t^2}\right) = \operatorname{Tr}_{\mathbb{C}l(1)}\left(We^{-(A+B\sigma)}\right) = \operatorname{Tr}\left(We^{-(A+B)}\right)^{even}$$

The term $W^\Delta e^{-(A^\Delta + B^\Delta)}$ is diagonal. Its trace is just twice the trace of $We^{-(A+B)}$.
On the coverings we will have the $\mathbb{C}l(1)$-Bismut superconnection $\tilde{\mathbb{B}}$ which is constructed in section A.3 and can be written explicitly (rescaled one)

$$\tilde{\mathbb{B}}_t = \sqrt{t}\tilde{\mathcal{D}}^\Delta\sigma + \tilde{\tilde{\nabla}}^\Delta - \frac{c(\tilde{T})^\Delta}{4\sqrt{t}}\sigma$$

The Γ-$\mathbb{C}l(1)$-supertrace on the coverings has a natural definition fibrewise. Moreover $\operatorname{Ker}\mathcal{D}^\Delta = \operatorname{Ker}\mathcal{D}\oplus\operatorname{Ker}\mathcal{D}$, and also spectral projections are obtained as before. In the computations of sections 5 and 7 about convergence of the $t\to\infty$-asymptotic in the even dimensional case we did not use that $\mathcal{D}$ was odd $\mathbb{Z}_2$-graded at all. This allows us to repeat here (when $\mathcal{D}$ is ungraded) all proofs without additional hypothesis and obtain proposition 7.15 also for the odd dimensional case.

8.1 Conjecture about applications to families of signature operators

In the direction of results of Keswani and Piazza-Schick about the homotopy invariance of the Cheeger-Gromov rho-invariant for the signature operator (recalled as **(b.2)** in section 1), one could think to apply proposition 7.15 to a family of signature operators (if one can prove that there exist examples of families for which regularity assumptions (1) and (2) above hold) and conjecture that, under suitable assumptions on the group Γ, the $\hat{\rho}$-form is a fiberwise homotopy invariant. Let's define a suitable notion of fibrewise homotopy equivalence between two fibrations.

DEFINITION 8.17 — Let $\pi : M \to B$ and $\pi' : N \to B$ be two fibrations of compact manifolds. A map $h : N \to M$ s.t. $\pi \circ h = \pi'$ is called a *fibrewise homotopy equivalence* if $\forall t \in B$ $h_{|N_t} : N_t \to M_t$ is a homotopy equivalence.

Let $\pi : M \to B$ a fibration of oriented Riemannian manifolds and let $\mathcal{D}$ be the family of the signature operators along the fibres. Let Γ be group and $R : M \to B\Gamma$ a map classifying a normal Γ-covering of M. Let $\tilde{\mathcal{D}}$ be the family of operators lifted to the coverings. Since the Novikov-Shubin invariants are Γ-homotopy invariants, we make the following remark.

REMARK — Suppose that the family $\tilde{\mathcal{D}}$ satisfies conditions (1) and (2) of proposition 7.15. Let $h : N \to M$ be a smooth fibrewise homotopy equivalence and denote with $\tilde{\mathcal{D}}'$ the family of signature operators on the fibres of $N \to B$. Then also the family $\tilde{\mathcal{D}}'$ satisfies conditions (1) and (2) of proposition 7.15.

CONJECTURE 8.18 — Let $\pi : M \to B$ a fibration of oriented Riemannian manifolds and suppose the family $\mathcal{D}$ of the signature operators along the fibres satisfies conditions (1) and (2) of proposition 7.15. Assume Γ is a torsion-free group that satisfies the Baum-Connes conjecture for the maximal C^*-algebra. Let $R : M \to B\Gamma$ a map classifying a normal Γ-covering of M. Then $\hat{\rho}_{(2)}(\tilde{\mathcal{D}}, r : M \to B\Gamma) = \hat{\rho}_{(2)}(\tilde{\mathcal{D}}', r \circ h : N \to B\Gamma)$.

Possible direction in the proof of Conjecture 8.18 The proof of **(b.2)** by Piazza and Schick in [PS1] makes use of the homotopy invariance of the Mishchenko-Fomenko index class in $K_0(C^*\Gamma)$ for the signature operator. The index class linked to our problem is an element of $K_0(C(B) \otimes C^*\Gamma)$. We only sketch briefly its definition in the case B is compact, following for example construction and notations of [MR].

Let $\pi : M \to B$ a fibration and let $R : M \to B\Gamma$ a map classifying a normal Γ-covering of M. Suppose for the moment that the Clifford modules bundle is $\mathbb{Z}_2$-graded, and the family of Dirac-type operators is odd. Let $\mathcal{V} = \tilde{M} \times_\Gamma C^*\Gamma \to M$ be the Mishchenko-Fomenko bundle on M. Its restriction on each fibre M_t gives $\mathcal{V}_{|M_t} = \tilde{M}_t \times_\Gamma C^*\Gamma \to M_t$. Twisting the family $\tilde{\mathcal{D}}$ with the Mischenko-Fomenko bundle we get a family that we denote $\mathcal{D}_\mathcal{V} = (D_{\mathcal{V},t})_{t\in B}$. $C^\infty(M/B, E \otimes \mathcal{V})$ has a $C(B) \otimes C^*\Gamma$-valued scalar product. Completing we get a $C(B) \otimes C^*\Gamma$-Hilbert module $H = L^2(M/B, E \otimes \mathcal{V})$.

Let $\mathcal{Q}$ a parametrix for the family $\mathcal{D}_\mathcal{V}^*\mathcal{D}_\mathcal{V}$, so that $\mathcal{Q}^2\mathcal{D}_\mathcal{V}^*\mathcal{D}_\mathcal{V} - I$ is a smooth family of smoothing operators. Consider the tern $\left[H, \mu, \begin{pmatrix} 0 & \mathcal{Q}\mathcal{D}_\mathcal{V}^* \\ \mathcal{D}_\mathcal{V}\mathcal{Q} & 0 \end{pmatrix}\right] \in KK_B(C(M), C(B) \otimes C^*\Gamma)$, where μ is given by multiplication. The map $c : C(B) \to C(M)$ given by inclusion of the functions constant along fibres gives a map $\varphi_c : KK_B(C(M), C(B) \otimes C^*\Gamma) \to$

$KK_B(C(B), C(B) \otimes C^*\Gamma) \simeq KK(\mathbb{C}, C(B) \otimes C^*\Gamma)$. Then

$$\mathcal{I}\mathrm{nd}\,(\mathcal{D}_{\mathcal{V}}) = \varphi_c\left(\left[H, \mu, \begin{pmatrix} 0 & \mathcal{Q}\mathcal{D}_{\mathcal{V}}^* \\ \mathcal{D}_{\mathcal{V}}\mathcal{Q} & 0 \end{pmatrix}\right]\right)$$

is the index class.

The homotopy invariance of $\mathcal{I}\mathrm{nd}\,(\mathcal{D}_{\mathcal{V}})$ in the case of the family of signature operators should follow from Hilsum-Skandalis result ([HS]) in the case of foliation. The index class of the family $\tilde{\mathcal{D}}$, viewed as foliation, belongs to a certain C^*-algebra which is Morita equivalent to $C(B) \otimes C^*\Gamma$. It should be possible to identify the two index classes and to obtain the homotopy invariance of $\mathcal{I}\mathrm{nd}\,(\mathcal{D}_{\mathcal{V}})$. This would be the a fundamental component in the proof of the conjecture.

A Superconnections

The notion of superconnection, given by Quillen in [Q], was motivated by the problem of giving a local formula for the index of a family of Dirac operators using heat equations methods. The idea of superconnection, extended then by Bismut to the case of infinite dimensional bundles, gives the right way represent the Chern character of the index bundle.

A.1 Superconnection adapted to a family of operators

Even dimensional fibre Let $\pi : M \to B$ be a fibration with fiber a compact manifold F of even dimension. Let $T(M/B)$ be the vertical tangent bundle and $g_{M/B}$ be a metric on the vertical tangent.

Let $E \to M$ be a hermitian vector bundle of Clifford modules on $Cl(T^*M/B, g_{M/B})$, and call $c_m : Cl(T^*M/B, g_{M/B}) \to \mathrm{End}(E_m)$ the Clifford action. Denote $M_b := \pi^{-1}(b)$ and $E_b := E_{|M_b}$. $E = E^+ \oplus E^-$ is naturally $\mathbb{Z}_2$-graded. Assume E is given a connection ∇^E such that

$$[\nabla^E_X, c(\alpha)] = c(\nabla^{M/B}_X \alpha). \tag{33}$$

These data produce a family of Dirac operators $\mathcal{D} = (D_b)_{b\in B}$: explicitely, if $b \in B$ is fixed, let $E_b = E_{|M_b}$ be the restricition of the bundle on the fibre M_b. $\forall m \in M_b$ then $T^*_m M_b = T^*_m(M/B)$, so that restricting to M_b, we have $c_b : Cl(T^*_m M_b) \to \mathrm{End}((E_b))_m$. The composition

$$D_b = c_b \circ \nabla^{E_b} : \mathcal{C}^\infty(M_b, E_b) \to \mathcal{C}^\infty(M_b, E_b)$$

gives the family of operators.

Bismut introduced some infinite dimensional fibre bundles associated to the family. Let's recall their definition: $\mathcal{E} \to B$ is the bundle with fibre $\mathcal{E}_b = \mathcal{C}^\infty(M_b, E_b)$. Its spaces of section are given by $\mathcal{C}^\infty(B, \mathcal{E}) = \mathcal{C}^\infty(M, q^*(\Lambda^* B) \otimes E)$.

DEFINITION A.19 — Let $\mathcal{D}$ a smooth family of Dirac operators on $E \to M$. A superconnection adapted to the family $\mathcal{D}$ is a differential operator $\mathbb{A}$ on $A(B, \mathcal{E}) = C^\infty(M, \pi^*\Lambda T^*M \otimes E)$ of odd parity s.t.

1. $\mathbb{A}(\nu\Phi) = (d_B\nu)\Phi + (-1)^{|\nu|}\mathbb{A}(\Phi)$
2. $\tilde{\mathbb{B}} = \tilde{\mathcal{D}} + \tilde{\mathbb{B}}_{[1]} + \ldots$,
   where $\tilde{\mathbb{B}}_{[i]} : \mathcal{A}^\bullet(B, W) \to \mathcal{A}^{\bullet+i}(B, W)$

Let now $\mathbb{A}$ be a superconnection adapted to the family of Dirac operators $\mathcal{D}$. Then $\mathbb{A}^2 =: \mathcal{F}$ is a family of differential operators with differential forms coefficients and moreover $\mathcal{F} = \mathcal{D}^2 + \mathcal{F}_{[+]}$, where $\mathcal{F}_{[+]}$ is a family of differential operators with differential forms coefficients which raise exterior degree in $\Lambda T_z^* B \otimes C^\infty(M_z, E_z)$.

For such a family $\mathbb{A}^2$ the heat kernel can be constructed ([BGV], chapter 9): it is denoted as $e^{-\mathbb{A}^2}$ and it is a family od smoothing operators along the fibres:

$$e^{-\mathbb{A}^2} \in C^\infty(B, \Lambda^* T^* B \otimes \mathcal{K}(E)) = \mathcal{A}(B, \mathcal{K}(E))$$

If $k \in \mathcal{K}(E)$ we have $\operatorname{Str} k \in C^\infty(B)$. If $k \in \mathcal{A}(B, \mathcal{K}(E))$ then $\operatorname{Str} k \in \mathcal{A}(B)$.

DEFINITION A.20 — Let $\mathbb{A}$ be a superconnection. The Chern character is defined as $\operatorname{ch} \mathbb{A} = \operatorname{Str} e^{-\mathbb{A}^2}$.

THEOREM A.1 —

1. $\operatorname{ch} \mathbb{A}$ *is a closed form on* B*;*
2. *if* $\mathbb{A}_s$ *is a family of superconnections adapted to* $\mathcal{D}_s$*, then*

$$\frac{d}{ds} \operatorname{ch} \mathbb{A}_s = -d_B \left(\operatorname{Str} \left(\frac{d\mathbb{A}_s}{ds} e^{-\mathcal{F}_s} \right) \right).$$

We now recall some results, explained for example in [BGV], chapter 9. Let $\mathcal{D}$ be the family of operators and $\mathbb{A}$ a superconnection adapted to $\mathcal{D}$. Then define, $\forall t > 0$, $\delta_t : \mathcal{A}(B, \mathcal{E})$ which on $\mathcal{A}^i(B, \mathcal{E})$ is multiplication by $t^{-\frac{i}{2}}$. Then $\mathbb{A}_t = t^{\frac{1}{2}} \delta_t \mathbb{A} \delta_t^{-1}$ is a superconnection adapted to the family $\sqrt{t}\mathcal{D}$.

Suppose now that the dimension of $\operatorname{Ker} D_z$ is constant, and let P_0 be the projection onto $\operatorname{Ker} \mathcal{D}$. Then $\nabla_0 = P_0 \mathbb{A} P_0$ is a connection on the index bundle $\operatorname{Ker} \mathcal{D} \to B$ and $\mathbb{A}_t = t^{\frac{1}{2}} \delta_t \mathbb{A} \delta_t^{-1}$ satisfies

$$\lim_{t \to \infty} \operatorname{ch}(\mathcal{A}_t) = \operatorname{ch}(\nabla_0)$$

moreover

$$\operatorname{ch} \nabla_0 - \operatorname{ch} \mathbb{A}_t = -d \int_t^\infty \operatorname{Str} \left(\frac{d\mathbb{A}_t}{ds} e^{-\mathbb{A}_s^2} \right) ds \tag{34}$$

A.2 Bismut superconnection and the Heat-equation proof of the local family index theorem

Explicit formula of the Bismut superconnection Let $\pi : M \to B$ be a fibration with fiber a compact manifold F of *even dimension*. Let $T(M/B)$ be the vertical tangent bundle. A connection on TM will be the choice of a subbundle $T^H M \subset TM$ s.t. $T^H M \oplus T(M/B) = TM$. When a connection is given, we will denote with $\mathcal{V}$ and $\mathcal{H}$ the projections on $T^H M$ and $T(M/B)$ relative to the splitting. When X is a section of TB, let X^H denote the unique section of $T^H M$ s.t. $\pi_* X^H = X$.

Let $g_{M/B}$ be a metric on the vertical tangent. Fix also any metric g_B on the base (and lift it on the horizontal bundle). By assuming that $T^H M$ and $T(M/B)$ are orthogonal, TM is endowed with the metric $g = \pi^* g_B \oplus g_{M/B}$. Let ∇^g the Levi-Civita connection on M with respect to the metric $\pi^* g_B \oplus g_{M/B}$. Define

$$\nabla^{M/B} = \mathcal{V} \nabla^g \mathcal{V}$$

on the vertical tangent: it can be shown that $\nabla^{M/B}$ does not depend on g_B. If ∇^B is the Levi-Civita connection on the base, consider also a second connection on TM, defined as

$$\nabla^{\oplus} = \nabla^{M/B} \oplus \nabla^B$$

which is compatible with the metric g, but is not torsion-free. Define the tensor

$$J = \nabla^g - \nabla^{\oplus}$$

and

$$T(U,V) = -J(U,V) + J(V,U), \tag{35}$$

the torsion of $\nabla^{\oplus}$. Define ω as

$$\omega(U)(X,Y) = g(J(U)(X),Y) \tag{36}$$

Define also Ψ as:

$$< \Psi(X,\theta), Z > = < \nabla^{M/B} X - \mathcal{V}[Z,X], \theta > \tag{37}$$

where θ is a section of $T^*(M/B)$, X is a section of $T(M/B)$ and $Z$is a section of $T_H M$. Let now $k(Z)$ be the trace of $\Psi(Z)$, i.e.

$$k(Z = \operatorname{tr}\Psi(Z) = \sum_i < \Psi(e_i, e^i), Z > \tag{38}$$

Let $\Omega \in C^\infty(M, \mathrm{Hom}(\Lambda^2 T_H M, T(M/B)))$ given by

$$\Omega(X,Y) = -\mathcal{V}[X,Y] \tag{39}$$

Let $E \to M$ be a hermitian vector bundle of Clifford modules, ∇^E a Clifford connection and $\mathcal{D} = (D_b)_{b\in B}$ the family of operators, then the Bismut superconnection, as written in [BGV][3]

$$\mathbb{B} = \underbrace{\sum_i c(e^i)\nabla^E_{e_i}}_{\mathbb{B}_{[0]}} + \underbrace{\sum_\alpha dy^\alpha \wedge \left(\nabla_{f_\alpha} + \frac{1}{2}k(f_\alpha)\right)}_{\mathbb{B}_{[1]}} \underbrace{- \frac{1}{4}\sum_{\alpha<\beta} c(\Omega(f_\alpha, f_\beta)) dy_\alpha \wedge dy_\beta}_{\mathbb{B}_{[2]}} \tag{40}$$

An equivalent formula for the Bismut superconnection is the follwing, given in [BC1] and [BF]: let $\tilde\nabla$ the connection on the infinite dimensional bundle $\mathcal{E} \to B$ defined by

$$\tilde\nabla_U \xi = \nabla^E_{U^H}\xi$$

Let $\tilde\nabla^u$ be a corrected connection where

$$\tilde\nabla^u_U = \tilde\nabla_U - \frac{1}{2}M_U$$

where $M_U = \sum_i < J(e_i)e_i, U >$.

The Bismut superconnection is

$$\mathbb{B} = \underbrace{\mathcal{D}}_{\mathbb{B}_{[0]}} + \underbrace{\tilde\nabla^u}_{\mathbb{B}_{[1]}} - \underbrace{\frac{c(T)}{4}}_{\mathbb{B}_{[2]}} \tag{41}$$

[3]In [BGV] this construction of $\mathbb{B}$ is obtained this way: $\mathbb{B}$ is the Dirac operator on the vector bundle $\mathbb{E} = \pi^*\Lambda^* T^* B \otimes E$ w.r.t. a Clifford connection $\nabla^{\mathbb{E},0} = \nabla^{\oplus} + \frac{1}{2}m_0(\omega)$ and the action m_0 of the Clifford algebra $C_0(M)$ endowed with a degenerate metric.

and $c(T) = \sum_{\alpha-\beta} c(T(f_\alpha, f_\beta)) dy_\alpha \wedge dy_\beta$. The theorem of Bismut [Bi] has the following consequence:

Theorem A.2 — *If $\mathcal{D}$ is a family of dirac operators associated to a Clifford connection ∇^E such that* $\dim \operatorname{Ker} D_b$ *is constant, then the superconnection in 41 (the so called Bismut superconnection) satisfies:*

$$\lim_{t\to 0} \operatorname{ch} \mathbb{B}_t = \int_{M/B} \hat{A}(M/B) \operatorname{ch}(E/S) \tag{42}$$

moreover the term $\operatorname{Str}\left(\frac{d\mathbb{A}_t}{ds} e^{-\mathbb{A}_s^2}\right)$ *is integrable for $t \to 0$.*

The index formula for families is

$$\operatorname{ch}(\mathcal{I}\mathrm{nd}\ \mathcal{D}) = \int_{M/B} \hat{A}(M/B) \operatorname{ch}(E/S) \in H^*(B) \tag{43}$$

A.3 Odd dimensional fibre and $\mathbb{C}l(1)$-superconnection

Suppose now that the fibres are odd dimensional: the bundle E is no more naturally $\mathbb{Z}_2$-graded. nevertheless there is a way to define also in this case a "superconnection" ([Q], [BC1])

Let $\mathbb{C}l(1) = \mathbb{C} \oplus \mathbb{C}\sigma$ the complexified Clifford algebra over the euclidean space $\mathbb{R}$, where σ is the generator s.t. $\sigma^2 = 1$. $\mathbb{C}l(1)$ is a noncommutative superalgebra.

Remark — A superbundle $V = V^0 \oplus V^1$ is a module over the algebra $\mathbb{C}l(1)$ if and only if it is of the form $V = W \oplus W$, and $\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. V is considered as a right module. The endomorphisms of V as $\mathbb{C}l(1)$-modules are of the form $\begin{pmatrix} F & G \\ G & F \end{pmatrix} = F\,\mathrm{Id} + G\sigma$, where $F, G : W \to W$.

Consider $E \oplus E$ and $\mathcal{D}$ the diagonal operator. Also $\tilde{\nabla}$ and $c(T)$ are considered as doubled diagonal operators. Consider the superconnection

$$\mathbb{B} = \mathcal{D}\sigma + \tilde{\nabla} - \frac{c(T)}{4\sqrt{t}}\sigma \tag{44}$$

which is odd w.r.t. the total grading and commutes with σ. This is known as the *Bismut* $\mathbb{C}l(1)$-*superconnection.*

B Γ-Hilbert spaces and Von Neumann dimensions

Let Γ be a discrete countable group. $l^2(\Gamma)$ is the completion of the pre-Hilbert space

$$\left(\mathbb{C}\Gamma,\ <(a_\gamma, b_\gamma)> := \sum a_\gamma \bar{b}_\gamma\right)$$

An element of $l^2(\Gamma)$ is represented by a formal sum $\sum_{g\in\Gamma} a_g g$, for complex numbers a_g s.t. $\sum_{g\in\Gamma} |a_g|^2 < \infty$. There is a unitary action L of Γ on $l^2(\Gamma)$, induced by left multiplication on

Γ. We can write it also in the following way, using the convolution product on the algebras $\mathbb{C}\Gamma$ and $l^2(\Gamma)$:

$$(f_1 * f_2)(\gamma) = \sum_{\sigma k=\gamma} f_1(\sigma) f_2(k)$$

The action L is given by

$$\begin{aligned} L : \Gamma &\to \mathcal{B}(l^2(\Gamma)) \\ \gamma &\mapsto \delta_\gamma * =: L_\gamma \end{aligned}$$

Observe that $\delta_\gamma * \delta_\beta = \delta_{\gamma\beta}$ and that $(L_\gamma f)(x) = (\delta_\gamma * f)(x) = f(\gamma^{-1}x)$. Convolution on the right gives in the same way a right action $R_\gamma(f) = f * \delta_\gamma$.

Definition B.21 — The *The Group Von Neumann algebra* $\mathcal{N}\Gamma$ is defined to be

$$\mathcal{N}\Gamma = \mathcal{L}(\Gamma) := \overline{L(\mathbb{C}\Gamma)}^{weak} \quad \text{in } \mathcal{B}(l^2(\Gamma)) \tag{45}$$

Correspondingly $\mathcal{R}(\Gamma)$ is the completion of $R(\mathbb{C}\Gamma)$ and by the double commutant theorem $\mathcal{L}\Gamma = \mathcal{R}\Gamma' = R(\mathbb{C}\Gamma)'$, so that $\mathcal{N}\Gamma$ is the algebra of operators which commute with the action of Γ.

An important feature of the group Von Neumann algebra is its *standard trace*

$$\begin{aligned} \mathrm{tr}_\Gamma : \mathcal{V}\Gamma &\to \mathbb{C} \\ A &\mapsto < Ae, e >_{l^2(\Gamma)} \end{aligned}$$

where e s the unit element. In particular for $A = \sum a_\gamma L_\gamma \in \mathcal{N}\Gamma$, then $\mathrm{tr}_\Gamma(A) = a_e$.

Definition B.22 — A *free Γ-Hilbert space* is a Hilbert space of the form $W \otimes l^2(\Gamma)$, where W is a Hilbert space and Γ acts on $l^2(\Gamma)$ on the right.
A *Γ-Hilbert space* H is a Hilbert space with a unitary action of Γ on the right s.t. there exists a Γ-equivariant immersion

$$H \to \mathcal{V} \otimes l^2(\Gamma)$$

in some free Γ-Hilbert space. For $\mathcal{H}_1, \mathcal{H}_2$ Γ-Hilbert spaces, define

$$\mathcal{B}_\Gamma(\mathcal{H}_1\mathcal{H}_2) := \{T : \mathcal{H}_1 \to \mathcal{H}_2 \ \text{ bounded and } \ \Gamma\text{-equivariant}\}$$

Observe that $\mathcal{B}_\Gamma(\mathcal{V} \otimes l^2(\Gamma)) \simeq \mathcal{B}(\mathcal{V}) \otimes \mathcal{L}\Gamma$.

B.1 Dimension theory

A bounded operator $T : \mathcal{H} \to \mathcal{H}$ on a Hilbert space is called *positive* if $< Tx, x >$ is real and $< Tx, x > \geq 0 \ \forall x \in \mathcal{H}$ (in particular, it is selfadjoint).

Let $\mathcal{H} = \mathcal{V} \otimes j^2(\Gamma)$ be a free Γ-Hilbert space. There exist a trace on $\mathcal{B}(\mathcal{H})_+$ with values in $[0, \infty]$: if $f \in \mathcal{B}(\mathcal{H})_+$.

$$\mathrm{tr}_\Gamma(f) = \sum_{j\in\mathbb{N}} < f(\psi_j \otimes \delta_e), \psi_j \otimes \delta_e >$$

if $(\psi_j)_{j\in\mathbb{N}}$ is a orthonormal base of $\mathcal{V}$ (tr_Γ is independent of the choice of the v=base.)

A Γ-trace can be defined also on any Γ-Hilbert-space H using the immersion $j : H \hookrightarrow \mathcal{V} \otimes l^2\Gamma$ and proveing that the trace does not depend on the choice of j (see [CG] or [Lu], pag. 17).

DEFINITION B.23 — (*Von Neumann dimension*) Let $\mathcal{H}$ be a Γ-Hilbert space, define

$$\dim_\Gamma(\mathcal{H}) = \mathrm{tr}_\Gamma(\mathrm{id} : \mathcal{H} \to \mathcal{H}) \quad \in \; [0, +\infty]$$

The Von Neumann dimension can take any non negative real number or ∞ as value.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Γ-Hilbert spaces. We describe some particular subspaces in $\mathcal{B}_\Gamma(\mathcal{H}_1, \mathcal{H}_2)$.

DEFINITION B.24 — The set of Γ-*finite rank operators* is

$$\mathcal{B}^f_\Gamma(\mathcal{H}_1, \mathcal{H}_2) := \{A \in \mathcal{B}_\Gamma(\mathcal{H}_1, \mathcal{H}_2) s.t. \dim_\Gamma \left(\overline{\mathrm{Im}\, A}\right) < \infty\}$$

The Γ-*compact operators* are

$$\mathcal{B}^\infty_\Gamma(\mathcal{H}_1, \mathcal{H}_2) := \overline{\mathcal{B}^f_\Gamma(\mathcal{H}_1, \mathcal{H}_2)}^{\|\ \|}$$

The Γ-*Hilbert-Schmidt operators*

$$\mathcal{B}^2_\Gamma(\mathcal{H}) := \{A \in \mathcal{B}_\Gamma(\mathcal{H}) s.t. \mathrm{tr}_\Gamma\,(AA^*) < \infty\}$$

and last the Γ-*trace class*

$$\mathcal{B}^1_\Gamma(\mathcal{H}) := \mathcal{B}^2_\Gamma(\mathcal{H})\mathcal{B}^2_\Gamma(\mathcal{H})^*$$

Their main properties are:

- $\mathcal{B}^f(\mathcal{H}), \mathcal{B}^\infty(\mathcal{H}), \mathcal{B}^2(\mathcal{H}), \mathcal{B}^1(\mathcal{H})$ are ideals.
- $\mathcal{B}^f \subset \mathcal{B}^1 \subset \mathcal{B}^2 \subset \mathcal{B}^\infty$
- $A \in \mathcal{B}^i(\mathcal{H})$ if and only if $|A| \in \mathcal{B}^i(\mathcal{H})$ for $i = 1, 2, f, \infty$.

B.2 Spectral measure from a Γ-operator

We will need only a few notions. Let $\mathcal{H}$ be a Γ-Hilbert space and let $T : \mathrm{Dom}\, T \subseteq \mathcal{H} \to \mathcal{H}$ be a not necessarily bounded operator s.t. $R_\gamma \,\mathrm{Dom}\, T \subseteq \mathrm{Dom}\, T$ and $TR_\gamma = R_\gamma T$ (i.e. T is not bounded but has the property of the operators of $\mathcal{N}\Gamma = R(\mathbb{C}\Gamma)'$. If $T = T^*$, let E_T be the projection valued measure on $\mathbb{R}$ associated to T. For every Borel set U the image $\mathrm{Im}\, E_T(U)$ is a Γ-invariant subspace of H.

DEFINITION B.25 —

$$\mu_{T,\Gamma}(U) := \mathrm{tr}_\Gamma(E_T(U) = \dim_\Gamma\,(\mathrm{Im}\, E_T(U)) \tag{46}$$

defines a measure on $\mathbb{R}$ with support in $\mathrm{spec}\, T$. For any bounded Borel function $f : \mathbb{R} \to [0, +\infty)$ it holds

$$\mathrm{tr}_\Gamma\,(f(T)) = \int_\mathbb{R} f d\mu_{\Gamma,T} \tag{47}$$

C Manifolds of bounded geometry

DEFINITION C.26 — Let (N,g) be a Riemannian manifold. N is *of bounded geometry* if

1. it has positive injectivity radius $i(N,g)$;
2. the curvature R_N and all its covariant derivatives are bounded.

A hermitian vector bundle $E\to N$ is *of bounded geometry* if the curvature R^E and all its covariant derivatives are bounded.This notion can be characterized in normal coordinates with conditions on g, coordinate transformations and ∇ (see for example in [Ro] and [Va]).

C.1 UC^k spaces of sections, Sobolev spaces, elliptic operators

Define the following spaces of section of a bundle $E\to N$:

$$UC^\infty(N)=\{f:N\to\mathbb{C}\ : f\in C^\infty\ ,\ \left\|\nabla^k f\right\|\le c(k)\}$$
$$UC^\infty(N,E)=\{\sigma\in C^\infty(N,E)\ ,\ \left\|\nabla^k\sigma\right\|\le c(k)\}$$

The Sobolev spaces of sections are defined as $H^k(N):=\overline{C_c^\infty}^{\|\ \|_k}$ where $\|f\|_k:=\sum_{j=0}^k\left\|\nabla^j f\right\|_{L^2(N\otimes^j T^*N)}$, and in a similar way for sections of a bundle $E\to N$. If $E\to N$ is bounded geometry, then there exists a special partition of the unit Φ_j s.t.:

- $\operatorname{supp}\Phi_j\subset U_j$, where U_j has normal coordinates;
- the derivatives of Φ_j's in normal coordinates are uniformly bounded indipendently from j and from choice of coordinates.
- $\|\cdot\|_{H^s(N,E)}\sim\sum_{j\in\mathbb{N}}\|\varphi_j\cdot\|^2_{H^j}(U_j,\mathbb{C}^N)$

The Sobolev embedding property still holds: if $\dim N=n$, then for $s>\frac{n}{2}+k$ $H^s(N,E)\hookrightarrow UC^k(N,E)$.

DEFINITION C.27 — The algebra $U\operatorname{Diff}(N,E)$ is the algebra generated by $UC^\infty(N,\operatorname{End}E)$ and $\{\nabla^E_X\}_{X\in UC^\infty(N,TN)}$. If $P\in U\operatorname{Diff}(N,E)$, then $\forall s\in\mathbb{R}$ it extends to an continuos operator $H^s(N,E)\to H^{s-k}(N,E)$

DEFINITION C.28 — $P\in U\operatorname{Diff}(N,E)$ is called *uniformly elliptic* if its principal symbol

$$\sigma_{pr}\in UC^\infty(T^*N,\pi^*\operatorname{End}E)$$

is invertible out of an ϵ-neighborhood of $0\in T^*N$, with inverse section which can be uniformly estimated.

For a uniformly elliptic operator T, Gårding estimate still holds:

$$\|\varphi\|_{H^{s+k}(N,E)}\le c(s,k)\left(\|\varphi\|_{H^s}+\|T\varphi\|_{H^s}\right)\quad\forall s\in\mathbb{R}$$

DEFINITION C.29 — An continuos operator $T:C_c^\infty(N,E)\to(C_c^\infty(N,E))'$ with Schwarzt

kernel $[T]$ *has order* $k \in \mathbb{Z}$ if $\forall s \in \mathbb{R}$ it admits a bounded extension $H^s(N,E) \to H^{s-k}(N,E)$. Hence it is closable as unbounded operator on $L^2(N,E)$. An operator $T \in \mathrm{Op}^k(N,E)$ is *elliptic* if it satisfies Gårding inequality.

Let $T \in \mathrm{Op}^k(N,E)$ an elliptic simmetric operator. Denote again by T its closure, $T : H^k \to L^2$. We recall some properties of $f(T)$, for opportune f. Define

$$RC(\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{C} \ : \ f \text{ is continuos and } \left| (1+x^2)^{\frac{k}{2}} f(x) \right| < \infty \ \ \forall k \}$$

$RC(\mathbb{R})$ is a Frechet space. If T has for example order 1, then from Gårding,

$$\|f(T)\varphi\|_{H^l} \leq c(l) \sum_{i=0}^{l} \left\| T^i f(T)\varphi \right\|_{L^2} \leq c(l) \, \|\varphi\|_{L^2} \sum_{0}^{l} |x^i f|_\infty$$

it also holds $\forall k, l \in \mathbb{Z}$, $l \geq k$:

$$\|f(T)\psi\|_{H^l} \leq c(l,k) \sum_{i=0}^{l-k} \left\| T^i f(T) \right\|_{H^k} \leq c \, \|\psi\|_{H^k} \sum_{0}^{l-k} |x^i f|$$

hence the map

$$\begin{gathered} RC(\mathbb{R}) \to \mathcal{B}(H^k, H^l) \\ f \mapsto f(T) \end{gathered}$$

is continuos and also $RC(\mathbb{R}) \to \mathrm{Op}^{-\infty}(N,E)$ is continuos. Now let's make the following observation on the Schwartz kernel of $f(T)$, denoted by $[f(T)]$.

PROPOSITION C.30 — *Let $L = [\frac{n}{2}+1]$ and $l \in \mathbb{N}$. The map*

$$\begin{gathered} \mathrm{Op}^{-2L-l}(N,E) \to UC^l(N \times N, E \times E^*) \\ T \mapsto [T] \end{gathered}$$

is continuos.

COROLLARY C.31 — If $f \in RC(\mathbb{R})$, then $[f(T)] \in UC^\infty(N \times N, E \times E^*)$ and

$$\begin{gathered} RC(\mathbb{R}) \to UC^\infty(N \times N, E \times E^*) \\ f \mapsto [f(T)] \end{gathered}$$

is continuos.

D Analysis on Γ-coverings of a compact manifold

Let $p : \tilde{M} \to M$ a regular Γ-covering (the action of Γ on $\tilde{M}$ is fixed on the right) of a compact Riemannian manifold. A fundamental domain is an open subset $\mathcal{I} \subset \tilde{M}$ s.t. $\mathcal{I} \cdot \gamma \cap \mathcal{I}$, $\forall \gamma \neq e$ and $\tilde{M} \setminus \bigcup \mathcal{I} \cdot \gamma$ has zero measure.

Let $E \to M$ a vector bundle, and $\tilde{E} = p^* E$ the lift. The Hilbert space $L^2(\tilde{M}, \tilde{E})$ is a Γ-free Hilbert space in the sense of definition B.22, in fact the map

$$\begin{gathered} L^2(\tilde{M}, \tilde{E}) \to L^2(\mathcal{I}, \tilde{E}_{|\mathcal{I}}) \otimes l^2(\Gamma) \\ \xi \mapsto \textstyle\sum_{\gamma \in \Gamma} \xi \gamma_{|\mathcal{I}} \otimes \delta_{\gamma^{-1}} \end{gathered}$$

with $(\xi\gamma)(x) = \xi(x\gamma^{-1}) \cdot \gamma$, is an isomorphism.

We need a description of Γ-trace class operators on $L^2(\tilde{M}, \tilde{E})$.

PROPOSITION D.32 — *Let $A \in \mathcal{B}_\Gamma(L^2(\tilde{M}, \tilde{E}))$. It holds*

1. $A \in \mathcal{B}_\Gamma(L^2(\tilde{M}, \tilde{E}))$ *if and only if* $\chi_{\mathcal{I}}|A|\chi_{\mathcal{I}} \in \mathcal{B}^1(L^2(\mathcal{I}, E_{|\mathcal{I}}))$
2. $A \in \mathcal{B}_\Gamma(L^2(\tilde{M}, \tilde{E})) \Rightarrow \mathrm{tr}_\Gamma(A) = \mathrm{tr}(\chi_{\mathcal{I}} A \chi_{\mathcal{I}})$
3. *if A has continuos kernel $[A]$, then*

$$\mathrm{tr}\, A = \int_{\mathcal{I}} \mathrm{tr}_{\tilde{E}_x}([A](x,x))\, dx = \int_M \pi_* \,\mathrm{tr}_{\tilde{E}_x}([A](x,x))\, dx$$

We need this Lemma.

LEMMA D.33 — *Let $T \in \mathrm{Op}^k_\Gamma(\tilde{M}, \tilde{E})$ elliptic and selfadjoint. Then*

$$\begin{aligned} RC(\mathbb{R}) &\to \mathcal{B}^1_\Gamma(\tilde{M}, \tilde{E}) \\ f &\mapsto f(T) \end{aligned}$$

is continuos.

E Novikov-Shubin invariants

Let $\tilde{M}$ be a regular Γ-covering of a compact Riemannian manifold M. Let $\triangle_k = d\delta + \delta d$ be the Laplacian on the k-forms of Mt. Since $\triangle_k$ is essentially selfadjoint, it has a unique closed extension $\bar{\triangle}_k$. Let $E^{(k)}_\lambda$ be the spectral measure for $\bar{\triangle}_k$, so that $\bar{\triangle}_k = \int \lambda dE\lambda^{(k)}$. Let

$$N_k(\lambda) = \mathrm{tr}_\Gamma\, E^{(k)}_\lambda$$

where tr_Γ is the Von Neumann trace on the algebra of Γ-invariant operators on L^2-sections of $\Lambda^k\tilde{M}$. $E^{(k)}_\lambda$ is in fact the density of the spectral measure $\mu_{\Gamma,\bar{\triangle}_k}$ defined in (46). Let

$$b^{(2)}_k = N_k(+0) = \lim_{\lambda\to 0+} N_k(\lambda) = \mathrm{tr}_\Gamma(P_k)$$

where P_k is the projection onto the kernel of $\bar{\triangle}_k$, be the k-th L^2 Betti number.

Consider the Laplace transform of the density $N_k(\lambda)$, given by

$$\theta_k(t) = \mathrm{tr}_\Gamma(\exp(-t\bar{\triangle}_k)) = \int e^{-\lambda t} dN_k(\lambda) \tag{48}$$

There is a link between the asymptotic of $N_k(\lambda)$ as $\lambda \to 0$ and the one of $\theta(t)$ as $t \to \infty$. In fact it holds: define $N_k(\lambda) = 0$ for $\lambda < 0$. The following conditions are equivalent: (see [GS], appendix)

1. $N_k(\lambda) - \bar{b} \asymp \lambda^\beta$ as $\lambda \to 0+$
2. $\theta_k(t) - \bar{b} = \asymp t^{-\beta}$ as $t \to \infty$

If (1) holds, then β is called the k-th Novikov Shubin invariant. In general (1) does not hold, and the Novikov-Shubin invariant is the number

$$\bar{\beta}_k = \liminf_{\lambda\to 0+} \frac{\log(N_k(\lambda) - \bar{b}_k)}{\log\lambda} = \liminf_{t\to\infty} \frac{-\log(\theta_k(t) - \bar{b}_k)}{\log t} \tag{49}$$

F Proof of Lemma 2.1

LEMMA F.34 — *Let $\pi : M \to B$ be a fibration and let $\tilde{M} \to M$ be a regular Γ-covering of the total space M. The composition $q = \pi \circ p : \tilde{M} \to B$ is still a fibration, and the fibre $q^{-1}(b) = \tilde{M}_b$ is a Γ-covering of M_b.*

PROOF— It is trivial that $\tilde{M}_b$ covers M_b, since $p_{|\tilde{M}_b} : \tilde{M}_b \to M_b$ is the restriction of the covering p. Now we prove that $q : \tilde{M} \to B$ is locally trivial. Fix $b_0 \in B$ and let $F = M_{b_0}$. Since π is a fibration, then $\exists U$ open neighborhood s.t. $\exists$ a homeomorphism $\varphi : \pi^{-1}(U) \to U \times F$ s.t.

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "\pi"] & U \times F \arrow[dl, "\pi_1"] \\
U &
\end{tikzcd}$$

Chose also U contractible. Consider now $q^{-1}(U) \subset \tilde{M}$ and the covering $\tilde{V} := q^{-1}(U) \to \pi^{-1}(U)$. The composition $\varphi \circ \pi$ is a covering of the product $U \times F$. We will show that $\tilde{V}$ is homeomorphic to a product.

Chose a basepoint $f_0 \in \pi^{-1}(b_0)$, and $\tilde{m}_0 \in q^{-1}(U)$ such that $\tilde{m}_0 \in q^{-1}((b_0, f_0))$. Then the covering $\varphi \circ p$ corresponds to a subgroup

$$H = (\varphi \circ p)_* \left(\Pi_1(\tilde{V}, \tilde{m}_0)\right) \leq \Pi_1(U \times F, (b_0, f_0))$$

Now since $\Pi_1(U \times F, (b_0, f_0)) \simeq \Pi_1(F, f_0)$, H corresponds to a subgroup of $\Pi_1(F, f_0)$ which we still call H. This characterizes a covering $\alpha : \hat{F} \to F$ s.t. $\alpha_* \left(\Pi_1(\hat{F}, \tilde{f}_0)\right) = H$, for a suitably chosen basepoint $\tilde{f}_0 \in \hat{F}$. Consider the covering $\beta := \mathrm{id} \times \alpha : U \times \hat{F} \to U \times F$ and the diagram

$$\begin{tikzcd}
\tilde{V} \arrow[d, "p"] \arrow[dr, "\varphi \circ p"] & U \times \hat{F} \arrow[d, "\beta"] \\
\pi^{-1}(U) \arrow[r, "\varphi"'] \arrow[d, "\pi"] & U \times F \arrow[dl, "\pi_1"] \\
U &
\end{tikzcd} \tag{50}$$

Suppose that $\tilde{V}$ is connected: then the two coverings $\tilde{V}$ and $U \times \hat{F}$ must be isomorphic since

$$\beta_* \left(\Pi_1(U \times F, (b_0, \tilde{f}_0))\right) = \beta_*(\Pi_1(\hat{F}, \tilde{f}_0)) = \alpha_* \left(\Pi_1(F, f_0)\right) = H$$

Then $\exists \Psi : \tilde{V} \to U \times \hat{F}$ which is homeomorphism and makes the diagram commute

$$\begin{tikzcd}
\tilde{V} \arrow[r, "\Psi"] \arrow[dr, "\varphi \circ p"'] \arrow[dd, "q"'] & U \times \hat{F} \arrow[d, "\beta"] \\
& U \times F \arrow[dl, "\pi_1"] \\
U &
\end{tikzcd}$$

hence verifies $\pi_1 \circ \Psi = q$, and gives the trivialization. In the case when $\tilde{V}$ is not connected (for example take the composition of the fibration $S^1 \times S^1 \to S^1$ with the universal covering of the torus: the fibres are disconnected), then each connected component covers $U \times F$ and we have $\Pi_0(\tilde{V}) \simeq \Pi_0(\tilde{M}_{b_0})$. Then reasoning component by component the conclusion also holds. $\square$

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